MATHEMATICAL TEXTS

FOR COLLEGES

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AN INTRODUCTION TO THE CALCULUS

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PREFACE

The course contained in this book has been designed to solve a particular problem which has arisen at Yale University, and it is hoped that it may be of use in other institutions where similar conditions prevail. The problem in question is to secure for students who have completed trigonometry a course which will present in one year the best conception of modern mathematics.

The natural procedure is a short course in analytic geometry followed by a course in the calculus. There are, however, numerous difficulties if the standard textbooks on these subjects are used. Perhaps the most serious is that the usual textbooks on the calculus presuppose at least a semester's work in analytic geometry and are themselves intended to cover a full year's work; hence adaptation of them to a much shorter course is not very satisfactory. Furthermore, the traditional arrangement of topics, if followed in a short course, results in a lack of unity in the subject and has the still more undesirable effect of bringing the applications of differentiation and integration so late that there is not sufficient time to digest them.

For these and other reasons it was thought best to abandon the treatment of analytic geometry as a separate subject and to build up the course around the calculus. In abridging the calculus the primary aim has been to present the fundamental ideas without an excessive amount of technique. The standard applications, including most of the problems to be found in more extensive treatises, are given

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AN INTRODUCTION TO THE CALCULUS

CHAPTER I

CARTESIAN COÖRDINATES. THE STRAIGHT LINE

1. Introduction. A survey of the problems which can be solved by elementary mathematics (algebra, geometry, and trigonometry) shows that although they are large in number they form a rather restricted class. For example, the problems of algebra consist mainly in finding one or more unknown quantities by the solution of equations; and in plane geometry and trigonometry we are confined to the study of figures bounded by straight lines and circles.

In order to solve more difficult problems we may proceed in two ways: (1) by using new processes of reasoning and calculation; (2) by combining our algebra and geometry so that we can use them together to greater advantage. The new processes above referred to, and to be described later, belong to the branch of mathematics known as calculus. We shall begin with the second way of proceeding, which, when carried out to its fullest extent, forms the subject of analytic geometry.

The first step in forming a combination of algebra and geometry is to devise some algebraic means for describing the position of a point. This can be done in many different ways, but the discussion in this book will be

limited to plane geometry, and only the most widely used method of locating a point in a plane will be introduced.

2. Rectangular Coördinates. In order to describe the position of a point in a plane, two reference lines, $X'X$ and $Y'Y$, are chosen at right angles to each other. Their point of intersection, O, is called the *origin*. The line $X'X$ is called the *x-axis*, and $Y'Y$ is called the *y-axis*.

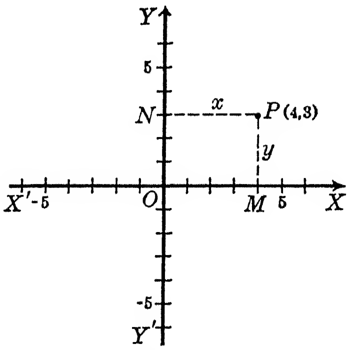


FIG. 1

The position of any point P in the plane is then described by giving its distance NP from the y-axis and its distance MP from the x-axis. In order to distinguish between points on opposite sides of the axes, the distance NP is regarded as positive if P is on the right of the y-axis, and negative if it is on the left; the distance MP is regarded as positive if it is measured upward from the x-axis and negative if it is measured downward.

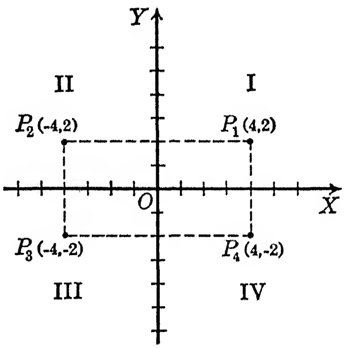


FIG. 2

Definitions. *The distance of a point from the y-axis is called the* ***abscissa****, or* ***x-coördinate****, of the point.*

The distance of a point from the x-axis is called the ***ordinate****, or* ***y-coördinate****, of the point.*

The abscissa is usually denoted by x, and the ordinate by y. The numbers x and y together are called the *coör-*

dinates of the point and are written (x, y). In Fig. 1, for example, $x=4$ and $y=3$; hence the coördinates of the point P are $(4, 3)$.

The four quadrants into which a pair of rectangular axes divides the plane are numbered as in trigonometry. If a point lies in the first quadrant, both coördinates are positive; if it lies in the second quadrant, its abscissa is negative and its ordinate is positive; if it lies in the third quadrant, both coördinates are negative; if it lies in the fourth quadrant, its abscissa is positive and its ordinate is negative.

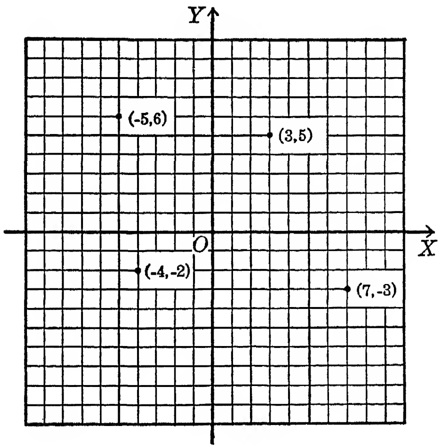


Fig. 3

Since the coördinates of two points not in the same quadrant cannot have the same sign, it follows that, after a pair of axes and a unit of length have been chosen, every point in the plane determines uniquely a pair of real numbers and, conversely, every pair of real numbers determines one and only one point in the plane.

3. Plotting. To "plot a point" given by rectangular coördinates is to mark it in its proper position, corresponding to the given coördinates. Thus, to plot the point $(3, 5)$ means to mark the point 3 units to the right of the y-axis and 5 units above the x-axis. In practice we measure 3 units to the right along the x-axis and then

5 units upward from the x-axis. Coördinate, or plotting, paper is made by ruling off equidistant lines parallel to the axes so that distances may readily be measured. The figure shows several points plotted on coördinate paper.

4. Symmetrical Points. Two points, A and B, are said to be symmetrical with respect to a line l if this line is the perpendicular bisector of the line AB. The line l is called the *axis of symmetry*. In Fig. 2 the points P_1 and P_2 are symmetrical with respect to the y-axis, and the points P_1 and P_4 are symmetrical with respect to the x-axis. It is evident that if l is the axis of symmetry of A and B, a rotation of the plane through 180° about l will interchange the points A and B.

Two points, A and B, are said to be symmetrical with respect to a point O, called the *center of symmetry*, if O is the mid-point of the line AB. In the figure referred to, P_1 and P_3 (also P_2 and P_4) are symmetrical with respect to the origin.

PROBLEMS

1. Plot accurately the points $(4, 3)$, $(4, -3)$, $(-3, 4)$, $(-3, -4)$, $(7, 0)$, $(0, -5)$, $(8, 0)$, $(0, 10)$.

2. Plot as accurately as possible the points $(2.3, 7.8)$, $(\sqrt{3}, -\sqrt{3})$, $(-1.6, -5.2)$, $(-\sqrt{5}, \sqrt{10})$.

3. Let A be the point $(4, 6)$. What are the coördinates of the point symmetrical to A with respect to the x-axis? the y-axis? the origin?

4. Three vertices of a square are $A(2, 0)$, $B(8, 0)$, $C(2, 6)$. What are the coördinates of the fourth vertex? What is the area of the square? What is the length of the diagonal?

5. Three vertices of a rectangle are $A(-2, -2)$, $B(-2, 6)$, $C(2, -2)$. What are the coördinates of the fourth vertex? What is the area of the rectangle? What is the length of a diagonal?

6. The vertices of a right triangle are $A(2, 1)$, $B(8, 1)$, $C(8, 7)$. What is the area of the triangle? What is the length of the hypotenuse?

7. Draw the triangles having the following points as vertices and calculate the area of each:

a. $(-2, 0), (6, 0), (4, 8)$.
b. $(0, 0), (6, 2), (0, 10)$.
c. $(0, 2), (12, 2), (5, -2)$.
d. $(-1, -3), (7, 7), (-1, 5)$.

8. A line joining two points is bisected at the origin. Find the coördinates of one end if the coördinates of the other end are

a. $(5, 3)$.
b. $(6, 4)$.
c. $(0, 4)$.
d. $(-10, 0)$.

9. Three vertices of a parallelogram are $(0, 0)$, $(a, 0)$, and (b, c). Find the coördinates of the fourth vertex. Prove your answer.

10. A square whose side has the length $2a$ has its center at the origin. What will be the coördinates of its vertices if (a) the sides are parallel to the axes? (b) the diagonals coincide with the axes?

11. An equilateral triangle whose side has the length a has its base on the x-axis. What are the coördinates of the vertices of the triangle if (a) the center of the base is at the origin? (b) one vertex is at the origin?

12. A regular hexagon whose side has the length a has its center at the origin and one diagonal along the x-axis. Find the coördinates of the vertices.

5. Directed Lines. The great generality of analytic methods and formulas is due primarily to the use of *directed lines.* These are lines upon which lengths are regarded as positive or negative according to the direction in which they are read. The positive direction can be assigned arbitrarily and is usually indicated by an arrowhead. If

the direction of a segment of a directed line is changed, its sign is changed; that is,

$$AB = -BA \quad \text{and} \quad BA = -AB.$$

FIG. 4

In adding segments of directed lines it is understood that the addition is to be performed *algebraically*. For example, in the figure

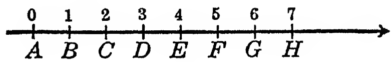


FIG. 5

$$AH = 7,\ HA = -7,\ DF = 2,\ GD = -3, \text{ etc.},$$
$$AC + CF = 2 + 3 = 5 = AF,$$
$$AC + CB = 2 + (-1) = 1 = AB,$$
$$GC + CE = -4 + 2 = -2 = GE.$$

The last three equations lead at once to the following theorem:

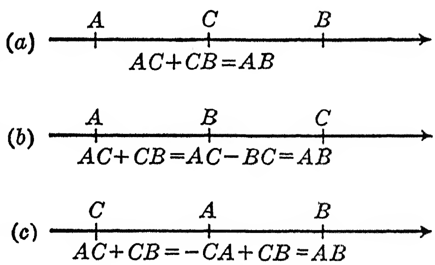


FIG. 6 (*a*), (*b*), (*c*)

Theorem. *If C is any point on the directed line passing through A and B, then $AC + CB = AB$.*

Proof. When AB is positive there are three cases, as indicated by the opposite figures.

Exercise. Write out the proofs for the three cases when AB is negative.

This theorem illustrates the advantage of the idea of directed lines. If one goes from a point A to a point C

and then from C to a point B, it requires no argument to show that the *net* result of the two motions is the same as going from A directly to B. The importance of the theorem lies in the fact that *one* equation (that is, $AC + CB = AB$) covers *all possible* cases. This equation is the same whether all the three motions are forward, or all backward, or one forward and two backward, etc.

Rule for Directed Lines in Analytic Geometry. *In using rectangular coördinates, unless the contrary is specifically stated, we shall assume that* (1) *the positive direction along the x-axis and along all lines parallel to the x-axis is from left to right;* (2) *the positive direction along all other lines is upward.*

The rules for the signs of coördinates given in § 2 agree with this, and hence an abscissa or an ordinate is simply a special case of length on a directed line.

6. Horizontal and Vertical Distances between Points. Let P_1 and P_2 be any two points, and let a line be drawn through P_1 parallel to the x-axis and meeting at Q the perpendicular from P_2 to the x-axis. The directed line P_1Q is called the *horizontal distance* (or distance parallel to the x-axis) from P_1 to P_2. The directed line QP_2 is called the *vertical distance* (or distance parallel to the y-axis) from P_1 to P_2.

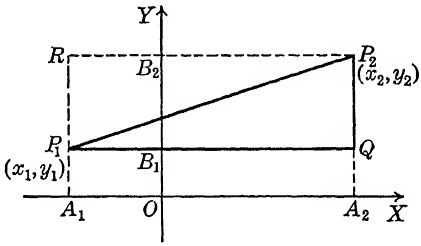


FIG. 7

The horizontal distance from P_1 to P_2 is evidently the same as the projection of P_1P_2 on the x-axis or any line parallel to the x-axis. Thus, in the figure, $P_1Q = A_1A_2 = RP_2$.

The vertical distance is evidently the same as the projection of P_1P_2 on the y-axis or any line parallel to the y-axis. Thus, in the figure, $QP_2 = B_1B_2 = P_1R$.

Theorem. *The horizontal distance from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ is $x_2 - x_1$; the vertical distance from P_1 to P_2 is $y_2 - y_1$.*

Proof. Using Fig. 7, the horizontal distance is P_1Q, which is equal to A_1A_2. But the theorem of § 5, on adding segments, gives

$$A_1A_2 = A_1O + OA_2.$$

Now $$x_1 = OA_1,$$

and $$x_2 = OA_2;$$

hence, by substitution, $A_1A_2 = x_2 - x_1$.

In like manner we find that

$$QP_2 = B_1B_2 = y_2 - y_1.$$

The proofs for the special cases in which the line P_1P_2 is parallel to one of the coördinate axes follow in the same way.

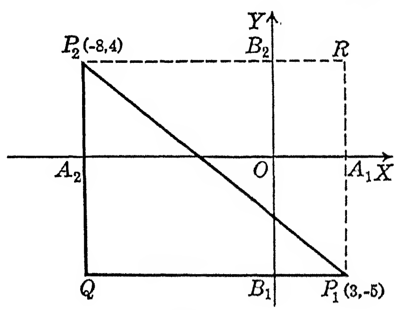


FIG. 8

It should be observed that any or all of the coördinates may be negative numbers. The truth of the theorem should be tested by marking several positions of the points P_1 and P_2. For example, in Fig. 8,

$$x_1 = 3,\ x_2 = -8,\ y_1 = -5,\ y_2 = 4.$$

Here $$P_1Q = A_1A_2 = (-8) - (3) = -11,$$

and $$QP_2 = B_1B_2 = (4) - (-5) = 9.$$

7. Distance between Two Points. ***Theorem.*** *The distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by the formula*

(I) $$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

Two figures are drawn in connection with the proof, which may be used without change of wording for *any* positions of the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

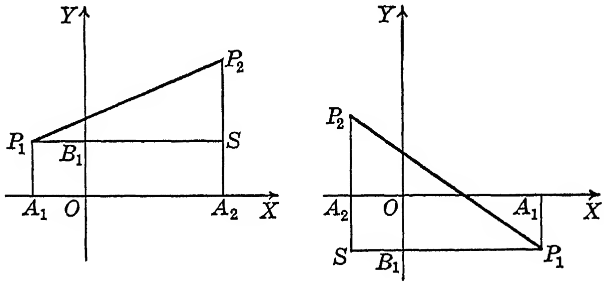


FIG. 9

Proof. Draw a line through P_1 parallel to OX, and one through P_2 parallel to OY; let these lines meet at S.

Now $$d = P_1P_2 = \sqrt{\overline{P_1S}^2 + \overline{SP_2}^2}.$$

But P_1S and SP_2 are the horizontal and vertical distances, respectively, from P_1 to P_2. Hence, by the theorem of § 6,

$$P_1S = x_2 - x_1 \quad \text{and} \quad SP_2 = y_2 - y_1.$$

Substituting these above, we have

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

8. The Mid-point Formulas. ***Theorem.*** *The coördinates of the point bisecting the line joining the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are given by the formulas*

(II) $$x=\frac{x_1+x_2}{2}, \qquad y=\frac{y_1+y_2}{2}.$$

Proof. Let P be the mid-point of the line joining P_1 and P_2. Since a straight line parallel to the base of a triangle and bisecting one side bisects the other and is equal to half of the base, we have, in either figure, $PC = P_1A$.

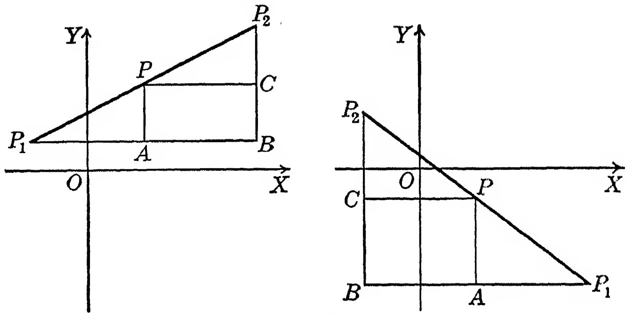


FIG. 10

But the theorem of § 6 gives

$$P_1A = x - x_1, \quad \text{and} \quad PC = x_2 - x.$$

Hence $$x - x_1 = x_2 - x.$$

Solving for x, we have $$x = \tfrac{1}{2}(x_1 + x_2).$$

Similarly, $AP = CP_2$, or $y - y_1 = y_2 - y$.

Hence $$y = \tfrac{1}{2}(y_1 + y_2).$$

EXAMPLE 1. Given the triangle $A(6, 0)$, $B(2, 4)$, $C(1, -1)$. Find the length of the median drawn from C.

Solution. It is important that an accurate figure should be drawn, and that the coördinates of the points should be marked on the figure. As results are calculated they should, as far as possible, be marked on the figure, and

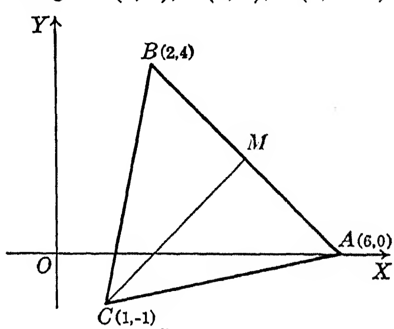


FIG. 11

all calculated results should be compared with the figure to see that they are reasonable.

The coördinates of M, the mid-point of AB, are given by the mid-point formulas:

$$x = \tfrac{1}{2}(2 + 6) = 4, \qquad y = \tfrac{1}{2}(4 + 0) = 2.$$

The length CM is given by the distance formula

$$CM = \sqrt{(4-1)^2 + (2+1)^2} = \sqrt{18} = 4.24.$$

This result can be checked by measurement by using a strip cut from the coördinate paper.

EXAMPLE 2. Prove analytically that the lines joining the mid-points of adjacent sides of any rectangle form a rhombus.

Solution. Let $ABCD$ be the given rectangle, with $AB = a$ and $AD = b$. Letters a and b are chosen to represent the sides in order that the following proof may be valid for *any* rectangle. If numerical values were used for a and b, it would be shown that for *one particular* rectangle the lines joining the mid-points of adjacent sides form a rhombus. But this proves nothing about any other rectangle. Hence it is necessary to give a proof with letters which represent *any* (that is, *every possible*) rectangle.

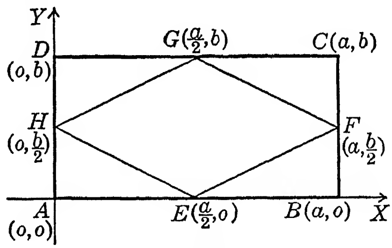


FIG. 12

To use analytic methods, coördinate axes must be introduced. Any set of axes could be used theoretically, but the most convenient will be those obtained by choosing the line AB for the x-axis and AD for the y-axis. The coördinates of the vertices will then be $A\ (0, 0)$, $B\ (a, 0)$, $C\ (a, b)$, and $D\ (0, b)$. The coördinates of the mid-points of the sides will be

$$E\left(\frac{a}{2}, 0\right), \quad F\left(a, \frac{b}{2}\right), \quad G\left(\frac{a}{2}, b\right), \quad H\left(0, \frac{b}{2}\right).$$

Application of the distance formula gives

$$EF = \sqrt{\left(a - \frac{a}{2}\right)^2 + \left(\frac{b}{2} - 0\right)^2} = \frac{1}{2}\sqrt{a^2 + b^2},$$

$$FG = \sqrt{\left(a - \frac{a}{2}\right)^2 + \left(\frac{b}{2} - b\right)^2} = \frac{1}{2}\sqrt{a^2 + b^2},$$

$$GH = \sqrt{\left(\frac{a}{2} - 0\right)^2 + \left(b - \frac{b}{2}\right)^2} = \frac{1}{2}\sqrt{a^2 + b^2},$$

$$HE = \sqrt{\left(\frac{a}{2} - 0\right)^2 + \left(0 - \frac{b}{2}\right)^2} = \frac{1}{2}\sqrt{a^2 + b^2}.$$

Since all four sides are equal, $EFGH$ is a rhombus.

9. Point dividing a Segment in a Given Ratio. Let P_1 and P_2 be two fixed points on a directed line. Any third point P divides the segment P_1P_2 into two segments P_1P and PP_2. The ratio of these two segments, denoted by r, is called the *ratio of division*. By definition,

$$r = \frac{P_1P}{PP_2}.$$

The ratio r is usually an integer or a common fraction, and its value is determined as in the following examples:

1. Let P be the mid-point of P_1P_2.

Then $$P_1P = PP_2$$

and $$r = 1.$$

2. Let P cut off one third of the segment P_1P_2.

Then $$P_1P = \tfrac{1}{2}\, PP_2$$

and $$r = \tfrac{1}{2}.$$

3. Let P be the point reached by extending P_1P_2 by half of its length.

Then $$P_1P = -\,3\, PP_2$$

and $$r = -\,3. \qquad \text{(Why is } r \text{ negative?)}$$

If the point of division P lies between P_1 and P_2, the two segments P_1P and PP_2 have the same sign, and r is positive. In this case P_1P_2 is said to be divided *internally*. If

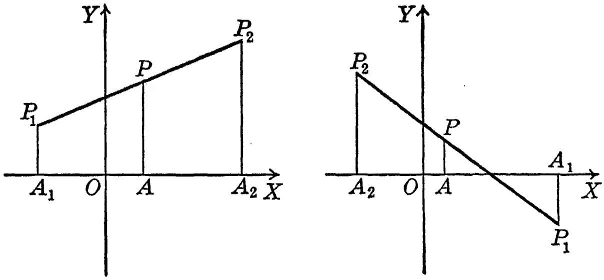


FIG. 13

P does not lie between P_1 and P_2, the two segments have opposite signs, and r is negative. In this case P_1P_2 is said to be divided *externally*.

Theorem. *If the segment P_1P_2 of the directed line passing through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is divided in the ratio r by the point $P(x, y)$, then*

$$\text{(II}a\text{)} \qquad x = \frac{x_1 + rx_2}{1+r}, \qquad y = \frac{y_1 + ry_2}{1+r}.$$

Proof. Draw the ordinates A_1P_1, AP, and A_2P_2.

Since the segments of two transversals comprehended between parallel lines are proportional, we have

$$\frac{P_1P}{PP_2} = \frac{A_1A}{AA_2}. \tag{1}$$

But $\quad A_1A = x - x_1 \quad$ and $\quad AA_2 = x_2 - x, \qquad$ by § 6

while $\quad \dfrac{P_1P}{PP_2} = r. \qquad$ By definition

Hence equation (1) becomes, by substitution,

$$\frac{x - x_1}{x_2 - x} = r.$$

5. Find the lengths of the diagonals of the parallelogram of Problem 4. Also show that these diagonals bisect each other.

6. Show that $(0, -3)$, $(7, 2)$, $(2, 9)$, and $(-5, 4)$ are the vertices of a square. Find its area.

7. Find the lengths of the medians and the center of gravity of the triangle $(5, -3)$, $(7, 5)$, $(2, 8)$.

8. Find the coördinates of the points which trisect the line joining $A(-2, 4)$ and $B(8, -8)$.

9. Given the quadrilateral $(2, -1)$, $(5, 6)$, $(3, 8)$, $(-4, 4)$.

a. Find the perimeter of the figure.

b. Show that the lines joining the middle points of opposite sides bisect each other.

10. If the middle point of a line is $(6, 2)$, and one end of the line is $(10, -4)$, what are the coördinates of the other end?

11. If $A = (-2, 6)$ and $B = (3, -1)$, and AB is prolonged to C, a distance equal to three times its length, find the coördinates of C.

12. Show analytically that the coördinates of the center of gravity of the triangle whose vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3) are $\frac{1}{3}(x_1 + x_2 + x_3)$ and $\frac{1}{3}(y_1 + y_2 + y_3)$.

13. Prove analytically that the diagonals of any rectangle are equal.

14. Prove analytically that the middle point of the hypotenuse of any right triangle is equidistant from the three vertices.

15. Prove analytically that the diagonals of any parallelogram bisect each other.

16. Prove analytically that the area of any triangle is four times the area of the triangle formed by joining the mid-points of its sides.

HINT. Take two vertices on the x-axis and the third on the y-axis.

17. Prove analytically that the distance between the mid-points of the nonparallel sides of any trapezoid is equal to half the sum of the parallel sides.

18. Prove that the area of the triangle whose vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3) is given by the formula $\pm \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3)$.

HINT. Circumscribe about the triangle a rectangle whose sides are parallel to the coördinate axes, and express the area of the given triangle as the difference between the area of the rectangle and three right triangles.

10. Angles; Inclination and Slope. ***Definition.*** *The angle between two directed lines is the angle between their positive directions.*

In either figure the angle between the directed lines OA and OB is the angle AOB.

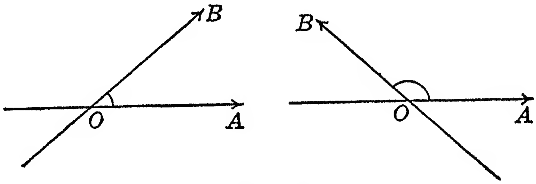


FIG. 14

Definition. *The inclination of a line is the angle between the line and the x-axis.*

The inclination of a line parallel to the x-axis is zero. Since the positive direction along all lines not parallel to

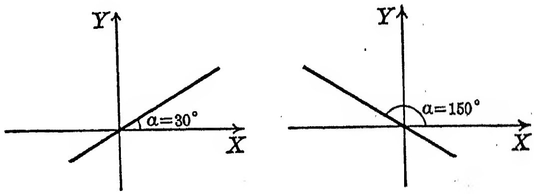


FIG. 15

the x-axis is upward, the inclination is the angle between the given line, directed upward, and the x-axis (or a line

parallel to the x-axis), directed toward the right. The inclination is taken always as a positive angle and is always less than 180°. It will be denoted by the letter α.*

Definition. *The slope of a line is the tangent of its inclination.*

The slope will be denoted by m and the definition of slope may be written $m = \tan \alpha$. When α increases from 0° to 90°, $\tan \alpha$ increases from 0 to ∞, and when α increases from 90° to 180°, $\tan \alpha$ increases from $-\infty$ to 0. Hence the slope may be any real number, positive or negative. If α is less than 90°, the slope is positive; if α is greater than 90°, the slope is negative.

11. The Slope Formula. ***Theorem.*** *The slope of the line passing through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by the formula*

$$\text{(III)} \qquad m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Proof. We exclude from the proof the special cases when P_1P_2 is parallel to one of the coördinate axes. If P_1P_2 is parallel to

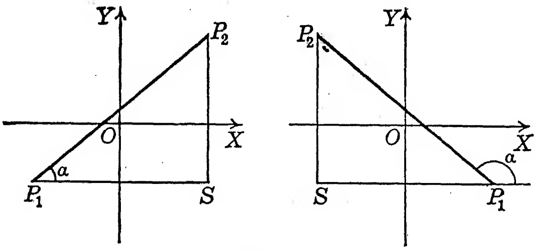


FIG. 16

the x-axis, the inclination is 0° and the slope $= \tan 0° = 0$. This agrees with the formula, since $y_2 = y_1$ and $x_2 \neq x_1$. If P_1P_2 is parallel to the y-axis, the inclination is 90° and the slope is infinite.

* This is the Greek letter *alpha*. For other Greek letters see the table of formulas at the end of the book.

Let the points be lettered so that P_1P_2 is the positive direction. Draw P_1S parallel to the x-axis, and SP_2 parallel to the y-axis. Two figures are possible, according as α is acute or obtuse. In either figure we have, by trigonometry,

$$m = \tan\alpha = \frac{SP_2}{P_1S}.$$

But $\qquad SP_2 = y_2 - y_1 \quad \text{and} \quad P_1S = x_2 - x_1.$ § 6

Hence
$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

It was assumed above that the points are lettered so that P_1P_2 is the positive direction. If the signs of the numerator and denominator in the formula are both changed, the value of the fraction is not changed, and we have the alternative form

$$m = \frac{y_1 - y_2}{x_1 - x_2}.$$

This shows that it is immaterial which point is called P_1.

The slope is the most convenient way of representing the direction of a line. Any two points on a line will give the same value for its slope. The formula should be thought of as the *vertical distance divided by the horizontal distance*, both distances being taken either from P_1 to P_2 or from P_2 to P_1.

12. To draw a Line with a Given Slope. If the slope of a line is positive, we can see that from a given point on the line it extends upward and to the right (or downward and to the left). If the slope is negative the line extends, from a given point on it, upward and to the left (or downward and to the right).

The slope formula provides us with a simple method for constructing a line passing through a given point and having a given slope. For example, to construct a line passing

through A (2, 3) and having a slope $\frac{1}{2}$, we measure from the point A a distance 2 units to the right and 1 unit upward, which brings us to the point B (4, 4). Then AB is the required line. This method of construction can be altered (1) by measuring any number of units to the right and then half as many units upward or (2) by measuring to the left and downward.

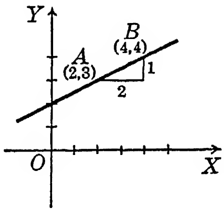


FIG. 17

The essential idea is that the slope of a line is the *rate of change* of the ordinate with respect to the abscissa; that is, the change in the ordinate per unit change in the abscissa. In the example above, the change in the y-coördinate is $\frac{1}{2}$ unit per unit change in the x-coördinate.

13. Parallel and Perpendicular Lines. *Theorem. If two lines are parallel, their slopes are equal, and conversely; if they are perpendicular, the slope of one is the negative reciprocal of the slope of the other, and conversely.*

Proof. Let α_1, α_2 and m_1, m_2 denote the inclinations and slopes, respectively, of the lines l_1, l_2.

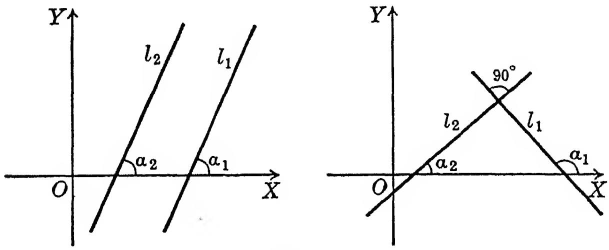


FIG. 18

If l_1 and l_2 are parallel, $\alpha_1 = \alpha_2$. Hence $m_1 = m_2$.

If l_1 and l_2 are perpendicular, let l_1 have the greater inclination

Then $$\alpha_1 = \alpha_2 + 90^\circ,$$

whence $$\tan \alpha_1 = \tan(\alpha_2 + 90^\circ).$$

Then, by trigonometry,

$$\tan \alpha_1 = -\cot \alpha_2 = -\frac{1}{\tan \alpha_2}.$$

Hence $$m_1 = -\frac{1}{m_2}, \text{ or } m_1 m_2 = -1.$$

The converse theorems are proved by retracing the above steps. For convenience in reference the theorems are restated as formulas.

Condition for parallelism:

(IVa) $$m_1 = m_2.$$

Condition for perpendicularity:

(IV b) $$m_1 m_2 = -1.$$

14. Angle between Two Lines. ***Theorem.*** *The angle θ between two directed lines l_1 and l_2 is given by*

(V) $$\theta = \alpha_1 - \alpha_2,$$

where α_1 is the greater inclination.

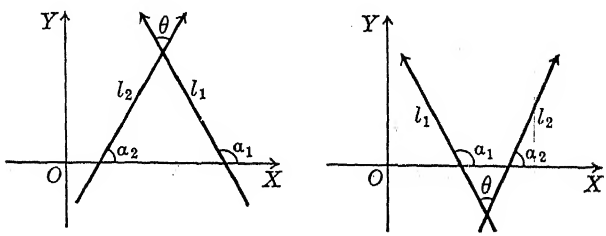


FIG. 19

Proof. In either figure, $\alpha_1 = \alpha_2 + \theta$,

whence $$\theta = \alpha_1 - \alpha_2.$$

The following theorem contains a formula which is important for certain problems:

Theorem. *If θ is the angle between two lines l_1 and l_2,*

$$\textbf{(V}a\textbf{)} \qquad \tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2},$$

where l_1 is the line of greater inclination.

Proof. By the above theorem,

$$\theta = \alpha_1 - \alpha_2.$$

Hence $\tan\theta = \tan(\alpha_1 - \alpha_2) = \dfrac{\tan\alpha_1 - \tan\alpha_2}{1 + \tan\alpha_1 \tan\alpha_2} = \dfrac{m_1 - m_2}{1 + m_1 m_2}.$

EXAMPLE 1. If the vertices of a triangle are $A(2, -1)$, $B(11, 1)$, $C(5, 8)$, prove that the median from B is perpendicular to AC.

Solution. By the mid-point formulas the coördinates of D, the mid-point of AC, are $(\frac{7}{2}, \frac{7}{2})$. Applying the slope formula, we obtain

$$\text{slope of } AC = \frac{8-(-1)}{5-2} = 3,$$

and

$$\text{slope of } BD = \frac{\frac{7}{2}-1}{\frac{7}{2}-11} = -\frac{1}{3}.$$

Since these slopes are negative reciprocals, the lines are perpendicular.

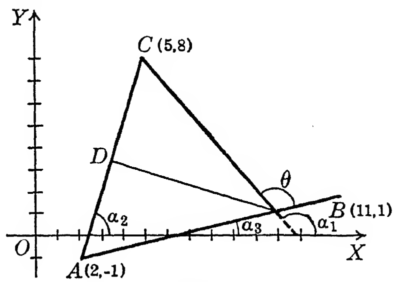


FIG. 20

EXAMPLE 2. Find the angles of the triangle ABC above.

Solution. Using the notation of the figure, we find

$$m_1 = \frac{8-1}{5-11} = -\frac{7}{6} = -1.1667, \text{ whence } \alpha_1 = 130.60° \; (130° \; 36');$$

$$m_2 = \frac{8+1}{5-2} = 3, \text{ whence } \alpha_2 = 71.57° \; (71° \; 34');$$

$$m_3 = \frac{1+1}{11-2} = \frac{2}{9} = 0.2222, \text{ whence } \alpha_3 = 12.53° \; (12° \; 32').$$

Applying Formula V, we obtain

$$A = \alpha_2 - \alpha_3 = 59.04°\,(59°\,2')$$

and

$$C = \alpha_1 - \alpha_2 = 59.03°\,(59°\,2').$$

At B, however, the angle between the positive directions of AB and BC is the exterior angle θ, which is supplementary to the angle ABC of the triangle. Here, then, $\theta = \alpha_1 - \alpha_3 = 118.07°$ $(118°\,4')$, and the required angle $B = 180° - \theta = 61.93°\,(61°\,56')$. As a check upon the work, note that $A + B + C = 180°$.

The angles may be found also by applying Formula Va, as follows:

$$\tan A = \frac{m_2 - m_3}{1 + m_2 m_3} = \frac{3 - \frac{2}{9}}{1 + \frac{2}{3}} = \frac{5}{3},$$

whence

$$A = 59.04°\,(59°\,2');$$

$$\tan C = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-\frac{7}{6} - 3}{1 + (-\frac{7}{2})} = \frac{5}{3},$$

whence

$$C = 59.04°\,(59°\,2');$$

$$\tan \theta = \frac{m_1 - m_3}{1 + m_1 m_3} = \frac{-\frac{7}{6} - \frac{2}{9}}{1 + (-\frac{7}{27})} = -\frac{15}{8},$$

whence

$$\theta = 118.07°\,(118°\,4'),$$

and

$$B = 180° - \theta = 61.93°\,(61°\,56').$$

Example 3. In the figure, $OBCA$ is a parallelogram. Find the coördinates of C.

Solution. Let the coördinates of C be (x, y). Since BC is parallel to OA, we have the slope of BC equal to the slope of OA, or

$$\frac{y - 2}{x - 5} = \frac{3 - 0}{1 - 0}. \qquad (1)$$

Similarly, since AC is parallel to OB,

$$\frac{y - 3}{x - 1} = \frac{2 - 0}{5 - 0}. \qquad (2)$$

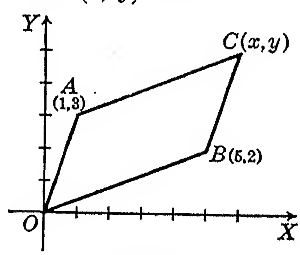


Fig. 21

Equations (1) and (2) are sufficient to determine x and y. We have, after simplification,

$$3x - y - 13 = 0,$$
$$2x - 5y + 13 = 0.$$

By solving these simultaneous equations for x and y, the result is found to be

$$x = 6, \quad y = 5.$$

Hence the coördinates of C are (6, 5).

PROBLEMS

1. Find the slope of the line joining

a. (3, 1), (7, 5).
b. (4, − 2), (2, 2).
c. (7, 3), (− 1, − 4).
d. (− 1, 6), (6, − 3).

2. What is the inclination of the line joining each of the following pairs of points? Check the results by measuring the inclinations with a protractor.

a. (2, 3), (7, 7).
b. (6, 1), (2, 4).
c. (5, − 2), (0, 4).
d. (2, − 3), (− 5, 7).

3. A point moves from left to right along each of the lines joining the following pairs of points. How much does it rise per horizontal unit? How far does it rise in moving from the second point to the point where $x = 15$? Explain the meaning of negative answers.

a. (3, − 2), (7, 8).
b. (1, 8), (4, 7).
c. (− 8, 3), (− 1, − 5).
d. (0, 0), (3, 11).

4. The slope of one line is 2.765, and that of another is − 1.370. Find the acute angle between them.

5. The slope of one line is − 0.784, and the inclination of another is 68.7° (68°42′). Find the acute angle between them.

6. Find the slope of a line which makes an angle of 30° with a line whose slope is 2.

7. Prove by means of slopes that $(-1, -2)$, $(5, 1)$, $(7, 2)$ lie on the same straight line.

8. Are $(-3, 5)$, $(7, 7)$, $(2, 1)$ the vertices of a right triangle? Prove your answer.

9. Are $(4, -2)$, $(16, 10)$, $(6, 19)$, $(-6, 8)$ the vertices of a rectangle? Prove your answer.

10. Find the angles of the following triangles:

a. $(1, 1)$, $(6, 3)$, $(3, 7)$.
b. $(2, 2)$, $(4, -6)$, $(8, 4)$.
c. $(3, 2)$, $(5, -5)$, $(-4, -2)$.
d. $(2, -3)$, $(-3, -2)$, $(-4, 1)$.

11. *a.* Prove that $(-3, 0)$, $(0, -3)$, $(4, 3)$, $(1, 6)$ are the vertices of a parallelogram.

b. Find the angles of the parallelogram.

c. Find the angle between the diagonals of the parallelogram.

12. *a.* Prove that $(-2, -1)$, $(1, -3)$, $(5, 3)$, $(0, 2)$ are the vertices of a trapezoid.

b. Find the angles of the trapezoid.

c. Find the angle between the diagonals.

13. *a.* Prove that $(0, -1)$, $(8, 3)$, $(5, 4)$, $(1, 2)$ are the vertices of an isosceles trapezoid.

b. Find the angles of the trapezoid.

c. Find the angle between the diagonals.

14. Three vertices of a rectangle are $(4, -2)$, $(16, 10)$, $(-6, 8)$. Find the fourth vertex.

15. The ends of the hypotenuse of an isosceles right triangle are $(7, -2)$ and $(-1, 4)$. Find the coördinates of the third vertex. How many solutions are there?

16. Three vertices of a parallelogram are $(-1, 2)$, $(3, 4)$, $(1, 7)$. What are the coördinates of the fourth vertex? How many possible solutions are there?

17. If $(2, -1)$, $(5, 5)$, $(-1, 3)$ are the middle points of the sides of a triangle, what are the coördinates of the vertices of the triangle?

18. The points $A(a, 0)$ and $B(a+b, 0)$ are any two points on the x-axis, and O is the origin. Two points C and D are taken in the first quadrant, and a third point E is taken in the fourth quadrant, so that OCA, ADB, and OEB are isosceles triangles having their bases on the x-axis and their base angles each 30°. Find the coördinates of C, D, and E.

HINT. $\tan 30° = \frac{1}{\sqrt{3}}$

19. Prove that the points C, D, and E in Problem 18 are the vertices of an equilateral triangle.

20. Prove analytically that the diagonals of a square are perpendicular.

21. Prove analytically that the lines joining the middle points of the sides of any quadrilateral, taken in order, form a parallelogram.

22. Prove analytically that in any triangle the line joining the middle points of two sides is parallel to the third side and equal to one half of it.

15. The Equation of a Straight Line. We have seen that to every point in the plane there corresponds, in rectangular coördinates, a single pair of numbers, and, conversely, that to every pair of numbers representing coördinates there corresponds a single point. We now extend the connection between algebra and geometry by showing that to every straight line in the plane there corresponds an algebraic equation of the first degree in x and y, and that, conversely, to every algebraic equation of the first degree in x and y there corresponds a straight line in the plane.

Definition. *The equation of a straight line is the equation satisfied by the coördinates of every point on the line and by those of no other point.*

Consider, for example, the straight line passing through the origin and making an angle of 45° with the x-axis.

Obviously, for any point on the line the abscissa is equal to the ordinate, and this is true for no other point, since points equidistant from the axes lie on the line. Hence the equation of the line is $x = y$. Similarly, the equation of the line passing through the origin with an inclination of 135° is $x = -y$.

It is not necessary that both x and y appear in the equation. The equation $x = 1$ is satisfied by the coördinates of every point on the line parallel to the y-axis and one unit to the right of it, and it is satisfied by the coördinates of no other point, since no other point can have its abscissa 1. The equation of any line parallel to the y-axis is obviously $x =$ a constant; the equation of the y-axis is $x = 0$. The equation of the x-axis is $y = 0$, and the equation of any line parallel to the x-axis is $y =$ a constant. The special case of lines parallel to the coördinate axes is omitted in the following theorem.

16. The Point-Slope Equation. ***Theorem.*** *The equation of the straight line passing through the point (x_1, y_1) and having the slope m is*

(VI) $$y - y_1 = m(x - x_1).$$

Proof. We are given a line l which has the slope m and passes through the point $P_1(x_1, y_1)$. We must show (*a*) that the coördinates of every point on l satisfy the above equation and (*b*) that the coördinates of no other point satisfy the equation.

(*a*) Let $P(x, y)$ be any point on the line l. Then the slope of P_1P is

$$m = \frac{y - y_1}{x - x_1}.$$

Hence $y - y_1 = m(x - x_1)$.

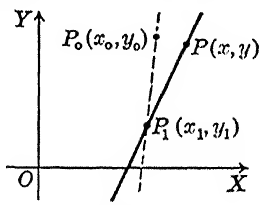


FIG. 22

(*b*) Let $P_0(x_0, y_0)$ be any point not lying on l. We must show that its coördinates cannot satisfy the equation just derived.

If they did, we should have

$$y_0 - y_1 = m(x_0 - x_1),$$

or

$$m = \frac{y_0 - y_1}{x_0 - x_1}.$$

This last relation states that m, the slope of l, equals the slope of the line through P_1 and P_0. But this is impossible, since two different lines through the same point cannot have the same slope.

Hence the coördinates of P_0 do not satisfy the equation.

The equation which we have derived above is known as the *point-slope form* of the straight-line equation. By means of it we can write down at once the equation of any straight line for which we can find the slope and the coördinates of one of its points.

EXAMPLE 1. Find the equation of the line which has an inclination of 135° and which passes through $(1, -3)$.

Solution. Since $\alpha = 135°$, we have $m = -1$. Substituting in Formula VI,

$$y + 3 = -1(x - 1),$$

and, simplifying, $x + y + 2 = 0.$

EXAMPLE 2. Find the equation of the line through $(2, -3)$ which is perpendicular to the line joining $A(3, 1)$ and $B(4, -5)$.

Solution. The slope of the line AB is

$$\frac{1+5}{3-4} = -6.$$

Hence the slope of the required line is $\frac{1}{6}$. Substituting in Formula VI,

$$y + 3 = \tfrac{1}{6}(x - 2);$$

and, simplifying, $x - 6y - 20 = 0.$

17. The Slope-Intercept Form. The *intercepts* of a line are the distances from the origin to the points where the line cuts the coördinate axes. Thus the x-intercept is the

abscissa of the point of intersection with the x-axis, and the y-intercept is the ordinate of the point of intersection with the y-axis.

The equation of the straight line which has the slope m and the y-intercept b is

$$\text{(VII)} \qquad y = mx + b.$$

This may be derived at once from the point-slope form, since we have given the slope and one point, $(0, b)$.

This form of the straight-line equation is called the *slope-intercept* form. Its most important use will be given in § 20.

18. Other Forms of the Straight-Line Equation. The point-slope form can be used in every case (except when the line is parallel to the y-axis and the equation is of the form $x = C$), but, as stated above, it is especially adapted to writing the equation of a line which is determined by a point and the direction of the line. Other special forms are convenient for other conditions determining a line. The slope-intercept form has already been found. Two others will be given here.

The Two-Point Form. The equation of the straight line passing through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$\text{(VIII)} \qquad \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

This is derived from the point-slope form by substituting for m its value $\frac{y_2 - y_1}{x_2 - x_1}$. It may also be obtained directly from a figure.

The Intercept Form. The intercepts of a line were defined in the previous section. The equation of the straight line whose x-intercept is a and whose y-intercept is b is

$$\text{(IX)} \qquad \frac{x}{a} + \frac{y}{b} = 1.$$

Proof. Substituting $x_1 = a$, $y_1 = 0$ and $x_2 = 0$, $y_2 = b$ in the two-point form, we have

$$\frac{y-0}{x-a} = \frac{b-0}{0-a}.$$

When simplified this becomes

$$-ay = bx - ba, \quad \text{or} \quad bx + ay = ab.$$

If each term is divided by ab, we obtain

$$\frac{x}{a} + \frac{y}{b} = 1.$$

EXAMPLE 1. Find the equation of the line passing through $(-1, 1)$ and $(2, 7)$.

Solution. Substituting in the two-point form,

$$\frac{y-1}{x+1} = \frac{7-1}{2+1},$$

and, simplifying, $\quad 2x - y + 3 = 0.$

An absolute check on the correctness of the result is furnished by the fact that two points determine a straight line. Hence the equation is correct if, and only if, it is satisfied by the coördinates of both points.

EXAMPLE 2. Find the equation of the line having the x-intercept 3 and the y-intercept -4.

Solution. Substituting in the intercept form,

$$\frac{x}{3} + \frac{y}{-4}$$

and, simplifying, $\quad 4x - 3y - 1$

PROBLEM

Find the equations of the lines determined by the following conditions. Draw the lines and check the answers.

1. Passing through $(-2, 3)$ and $(7, 9)$.

2. Passing through $(4, 1)$ and $(-2, 3)$.

3. Passing through $(1, 2)$ with slope 1

4. Passing through $(3, -4)$ with slope -3.

5. Passing through $(6, 0)$ and $(0, -4)$.

6. Passing through $(-2, 0)$ with inclination $45°$.

7. Passing through $(8, -4)$ with inclination $135°$.

8. Passing through $(2, -1)$ and parallel to the x-axis.

9. Passing through $(1, 4)$ and parallel to the y-axis.

10. Parallel to the y-axis and 7 units to the right of it

11. Parallel to the x-axis and 5 units below it.

12. Having the x-intercept -7 and the y-intercept 3.

13. Having the x-intercept -5 and the slope -2.

14. Having the x-intercept 7 and inclination $135°$.

15. Having the x-intercept -6 and passing through $(-2, 3)$

16. Having the y-intercept 2 and the slope $\frac{1}{2}$.

17. Having the y-intercept 7 and inclination $45°$.

18. Having the y-intercept -4 and passing through $(8, 6)$.

19. Passing through $(6, -8)$ and rising $\frac{1}{2}$ unit per unit increase in x.

20. Passing through $(-2, 5)$ and falling 2 units per unit increase in x.

19. The First-Degree Equation. It will now be shown, conversely, that to every equation of the first degree in x and y there corresponds a straight line in the plane. Every equation of the first degree in x and y can be written in the form $Ax + By + C = 0$. By the locus of an equation is meant the line (or curve) that contains all points whose coördinates satisfy the equation, and such points only.

Theorem. *The locus of the equation $Ax + By + C = 0$ is a straight line.*

Proof. Case I. If $B = 0$, then $x = -\dfrac{C}{A}$, and the locus is obviously a straight line parallel to the y-axis.

Case II. If $B \neq 0$, the equation may be solved for y, giving

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

Comparison with the slope-intercept form $y = mx + b$ shows that this is the equation of the straight line that cuts the y-axis at $\left(0, -\frac{C}{B}\right)$ and has the slope $m = -\frac{A}{B}$.

NOTE. It should be noted that the values of x and y that satisfy an equation are not changed if every term of the equation is multiplied by the same constant factor. That is, $Ax + By + C = 0$ and $kAx + kBy + kC = 0$ represent the same straight line.

20. To construct a Line, given its Equation. To draw a straight line when its equation is given, we find two points on the line. Any two points whose coördinates satisfy the equation will do, but the simplest to get are those where the line cuts the coördinate axes. Thus, to draw the line whose equation is $2x - 5y - 10 = 0$, we set $x = 0$, giving $y = -2$, and then set $y = 0$, giving $x = 5$. Hence two points on the line are $(0, -2)$ and $(5, 0)$. The straight line passing through these two points is the locus of the equation. If the line passes through the origin, another point not on the axes must be found. For accurate construction, this point should be as far from the origin as the plotting paper permits.

The slope of a line whose equation is given can be determined by inspection. It appears in the proof of the preceding theorem that the slope of the line whose equation is $Ax + By + C = 0$ is $m = -\frac{A}{B}$. This important fact can be stated also in the following form: *If the equation of a straight line is solved for y, the resulting coefficient of x is the slope.*

PROBLEMS

1. Find the intercepts of the following lines and draw the lines. Find also the slope and the inclination in each case.

a. $x - 2y + 6 = 0$. *c.* $2x - y - 10 = 0$.
b. $3x + 5y - 8 = 0$. *d.* $4x + 6y + 9 = 0$.

2. Find the angle between each of the following pairs of lines taken from Problem 1:

i. *a.* and *b.* iii. *a.* and *d.* v. *b.* and *d.*
ii. *a.* and *c.* iv. *b.* and *c.* vi. *c.* and *d.*

3. Find the equation of the line parallel to $x - y + 7 = 0$ and passing through the origin.

4. Find the equation of the line that is perpendicular to $2x - 3y + 5 = 0$ and that passes through the origin.

5. Find the equation of the line parallel to $x + 2y - 6 = 0$ and passing through $(-2, -3)$.

6. Find the equation of the line that is perpendicular to $4x - 5y + 9 = 0$ and that passes through $(2, 6)$.

7. Find the equations of the lines passing through the point $(-5, 4)$ and parallel to the lines in Problem 1.

8. Find the equations of the lines passing through the point $(5, -4)$ and perpendicular to the lines in Problem 1.

9. Given the triangle $A(3, -1)$, $B(-5, 5)$, $C(6, 5)$:

a. Find the equations of the sides.

b. Find the equations of the medians.

c. Find the equations of the perpendicular bisectors of the sides.

d. Find the equations of the lines joining the mid-points of the sides.

e. Find the equations of the lines through the vertices and parallel to the opposite sides.

f. Find the equations of the lines through the vertices and perpendicular to the opposite sides.

10. The equations of the sides of a triangle are $x + y - 5 = 0$, $2x - y + 4 = 0$, $2x - 5y - 20 = 0$.

a. Draw the triangle.

b. Find the slope and the inclination of each side.

c. Find the angles of the triangle.

11. The equations of two sides of a parallelogram are $3x - 5y + 15 = 0$ and $9x + 4y - 32 = 0$. Find the equations of the other two sides if one vertex is $(0, -5)$.

12. Find the equations of the sides of the square of which two opposite vertices are $(-2, 6)$ and $(8, 0)$.

13. A line of unknown slope m passes through the point $(2, 9)$ and forms with the coördinate axes a triangle of area 48. Find its slope and its equation.

HINT. Use the point-slope form, and find the intercepts in terms of m.

21. Points of Intersection. Let the equations of two lines be

$$A_1x + B_1y + C_1 = 0,$$
$$A_2x + B_2y + C_2 = 0.$$

Every point that lies on the first line will have coördinates that satisfy the first equation, and every point that lies on the second line will have coördinates that satisfy the second equation. Consequently the coördinates of the point of intersection of the two lines will satisfy both equations. Hence the coördinates of the point of intersection of two lines can be found by solving their equations simultaneously.

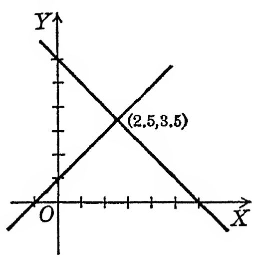


FIG. 23

EXAMPLE 1. Find the point of intersection of the lines whose equations are $x + y - 6 = 0$ and $x - y + 1 = 0$.

Solution. By solving the equations simultaneously, we get $x = 2.5$ and $y = 3.5$.

EXAMPLE 2. Find the center and the radius of the circumscribed circle of the triangle $A\,(3, 0)$, $B(1, 4)$, $C(-6, 3)$.

Solution. The center of the circumscribed circle is the point of intersection of the perpendicular bisectors of the sides.

The perpendicular bisector of AB passes through $F(2, 2)$ and has a slope that is the negative reciprocal of the slope of AB.

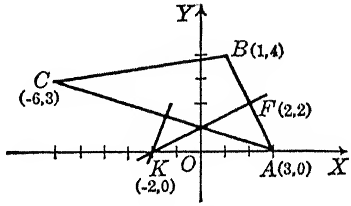


FIG. 24

Hence its equation is $y - 2 = \frac{1}{2}(x - 2)$,

or $$x - 2y + 2 = 0.$$

Similarly, the equation of the perpendicular bisector of AC is

$$y - \tfrac{3}{2} = 3\left(x + \tfrac{3}{2}\right),$$

or $$3x - y + 6 = 0.$$

The coördinates of the center of the circumscribed circle are found by solving these equations simultaneously. The result is $x = -2$, $y = 0$.

The radius of the circumscribed circle is the distance from the center K to any vertex of the triangle. As a check on the correctness of the work we shall find the distance to each of the vertices.

$$KA = 5 \text{ (from the figure)},$$
$$KB = \sqrt{(-2-1)^2 + (0-4)^2} = 5,$$
$$KC = \sqrt{(-2+6)^2 + (0-3)^2} = 5.$$

Since these three distances are the same, we have proved that K is equidistant from the three vertices.

The center of the circumscribed circle is $(-2, 0)$, and its radius is 5.

It is important in every problem of this type to draw the figure accurately and to make sure that the calculated results agree with the figure.

PROBLEMS

1. Find the point of intersection of the following lines and draw the figure:

a. $x - 2y + 4 = 0$ and $2x + y - 8 = 0$.
b. $2x + 3y - 8 = 0$ and $x - y + 1 = 0$.
c. $x + y - 3 = 0$ and $3x + 4y - 8 = 0$.
d. $3x - 4y - 12 = 0$ and $3x + 4y + 12 = 0$.
e. $x - 3y + 7 = 0$ and $2x + y - 8 = 0$.

2. Draw the triangle the equations of whose sides are $2x - 5y - 7 = 0$, $5x - y - 6 = 0$, and $17x + 15y - 94 = 0$, and find the coördinates of the vertices.

3. Draw the triangle the equations of whose sides are $x + y - 12 = 0$, $x - 2y + 3 = 0$, and $4x - y + 8 = 0$, and find the coördinates of the vertices.

4. Find the equations of the medians of the triangle $A(3, 3)$, $B(6, 2)$, $C(8, -2)$, and prove analytically that they meet in a point.

5. Find the equations of the perpendicular bisectors of the sides of the triangle $(7, 1)$, $(-1, -5)$, $(3, -7)$, and prove analytically that they meet in a point.

6. Find the center and the radius of the circle circumscribed about the triangle in Problem 4.

7. Find the center and the radius of the circle circumscribed about the triangle in Problem 5.

8. Find the center and the radius of the circle circumscribed about the triangle $(1, 3)$, $(-5, -2)$, $(7, 1)$.

9. Find the point of intersection of the diagonals of the quadrilateral $(5, 5)$, $(-6, 7)$, $(-7, -2)$, $(2, -4)$.

10. Three of the vertices of a parallelogram are $A(-6, 7)$, $B(-7, -2)$, and $C(2, -4)$, and ABC is one of its angles. Find the equations of the sides passing through the fourth vertex D and the coördinates of D.

4. If A, B, and C are fixed constants, the equation of the system of lines parallel to $Ax+By+C=0$ is $Ax+By+k=0$,

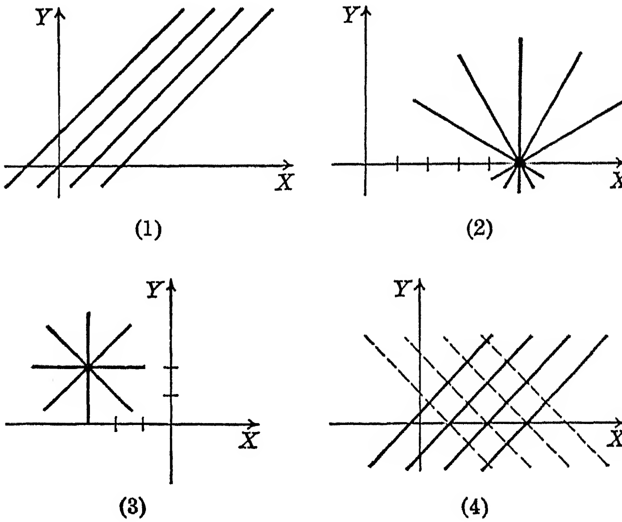


FIG. 25

where k is an arbitrary constant; and the equation of the system of lines perpendicular to the given line is $Bx-Ay+k=0$.

The equation of a system of lines is useful in the solution of certain types of problems, as illustrated below:

EXAMPLE 1. Find the equation of the line that passes through $(4, -2)$ and is parallel to the line $x+3y-10=0$.

Solution. The equation of the system of lines parallel to the given line is

$$x+3y+k=0. \tag{1}$$

If a line of this system passes through $(4, -2)$, its equation must be satisfied when $x=4$, $y=-2$; that is,

$$4-6+k=0, \text{ whence } k=2.$$

Substituting this value of k in (1) gives the final result

$$x + 3y + 2 = 0.$$

This method should be compared with the method of solving this problem without the use of the notion of a system of lines.

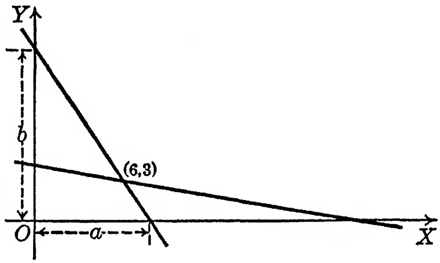


FIG. 26

EXAMPLE 2. Find the equation of the line that passes through (6, 3) and cuts off from the first quadrant a triangle of area 48 square units.

Solution. The equation of the system of lines passing through (6, 3) is

$$y - 3 = m(x - 6),$$

or

$$mx - y + 3 - 6m = 0. \tag{1}$$

The intercepts a and b of any one of these lines on the coördinate axes can readily be found in terms of m. We obtain

$$a = \frac{6m - 3}{m} \quad \text{and} \quad b = 3 - 6m.$$

The area of the triangle formed by the line and the coördinate axes is $\frac{ab}{2}$.

Hence

$$\frac{(3 - 6m)(6m - 3)}{2m} = 48,$$

which becomes, after simplification,

$$12m^2 + 20m + 3 = 0.$$

The two roots of this equation are

$$m = -\tfrac{3}{2} \quad \text{and} \quad m = -\tfrac{1}{6}.$$

Substituting in (1), we have the two results

$$3x + 2y - 24 = 0, \quad \text{and} \quad x + 6y - 24 = 0.$$

Both results are correct, as can be easily verified.

PROBLEMS

1. Write the equation of the system of lines determined by the following conditions:

a. Having the slope 2.

b. Making an angle of 30° with the x-axis.

c. Passing through $(-2, 3)$.

d. Having the intercept on the y-axis equal to 4.

e. Having the intercept on the x-axis equal to 2.

f. Having the intercept on the y-axis double the intercept on the x-axis.

g. Forming with the coördinate axes a triangle of area 8.

2. Determine k so that

a. The line whose equation is $x - ky + 3 = 0$ passes through $(3, 1)$.

b. The line whose equation is $x + 2y + k = 0$ passes through $(10, 1)$.

3. Using the same coördinate axes, draw several lines of the system represented by

a. $x - 2y + k = 0$.

b. $y = kx - 3$.

c. $kx + y + 2k = 0$.

d. $y - 4 = k(x + 2)$.

4. By inspection of the equations in Problem 3, determine the common geometrical property of each of the systems of lines there represented.

5. Write an equation that will represent the system of lines parallel to each of the following lines:

a. $x + 7y - 12 = 0$.

b. $5x - 2y + 14 = 0$.

c. $3x - 4y - 7 = 0$.

6. Write an equation which will represent the system of lines perpendicular to each of the following lines:

a. $x - 6y + 5 = 0$.

b. $4x + 2y - 9 = 0$.

c. $2x + 3y + 8 = 0$.

7. Find the equation of the line parallel to $3x - 5y + 8 = 0$ and passing through $(1, 4)$.

8. Find the equation of the line that is perpendicular to $2x + y - 3 = 0$ and that passes through $(7, 2)$.

9. The equations of two sides of a parallelogram are $3x - y + 7 = 0$ and $x + 2y - 5 = 0$. Find the equations of the other two sides if one vertex is $(3, -2)$.

10. Find the equation of the system of lines which are (*a*) parallel and (*b*) perpendicular to the line whose equation is $2x + 3y = 7$.

11. Determine k' in terms of k so that the line whose equation is $x + ky + k' = 0$ may pass through the point $(-3, 6)$.

12. Determine k and k' so that the line whose equation is $x + ky + k' = 0$ may pass through the points $(-3, 6)$ and $(6, 4)$.

23. Distance from a Line to a Point. To find the distance from a line parallel to the y-axis to a point involves no difficulty. If the equation of the line is $x = C$, and the

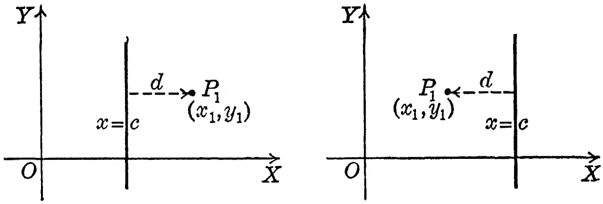


FIG. 27

coördinates of the point are (x_1, y_1), the distance *from* the line *to* the point is obviously $x_1 - C$, whether the point is at the right or at the left of the line.

For the more general case, where the line cuts the y-axis, we have the following theorem:

Theorem. *The distance from the line whose equation is $y = mx + b$ to the point $P_1(x_1, y_1)$ is given by the formula*

$$\textbf{(X)} \qquad d = \frac{y_1 - mx_1 - b}{\sqrt{m^2 + 1}}.$$

Proof. In either figure let l be the given line whose equation is $y = mx + b$. Through P_1 draw a line l' parallel to l, and through O draw ONN' perpendicular to l and consequently perpendicular to l'.

Obviously, by § 5,

$$d = NN' = ON' - ON. \tag{1}$$

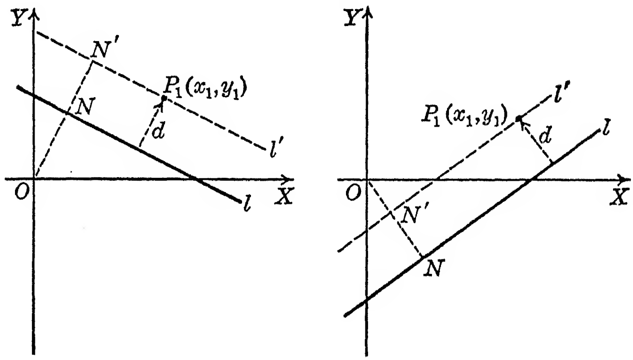


FIG. 28

Since ON is perpendicular to l, its slope is $-\frac{1}{m}$, and its equation is

$$y = -\frac{1}{m}x.$$

Solving this simultaneously with $y = mx + b$, we find that the coördinates of N are

$$x = \frac{-bm}{m^2+1}, \qquad y = \frac{b}{m^2+1}.$$

Substituting these in the distance formula, we have

$$ON = \frac{b}{\sqrt{m^2+1}}, \tag{2}$$

where the sign of the radical is $+$, since ON always has the same sign as b.

The equation of the line l' is

$$y - y_1 = m(x - x_1),$$

or

$$y = mx + y_1 - mx_1.$$

The y-intercept of this line is $b' = y_1 - mx_1$.

Hence equation (2) tells us that

$$ON' = \frac{y_1 - mx_1}{\sqrt{m^2+1}}. \tag{3}$$

Substituting the values of ON and ON' in equation (1) gives the required formula.

NOTE. If P_1 is on the line, Formula X gives $d = 0$, as should be the case. Since the proof has been based upon the idea of directed lines (equation (1)), it follows that NN' will be positive if it is directed upward, and negative if directed downward. Hence the value of d, as given by Formula X, will be positive if P_1 lies above the line, and negative if P_1 lies below the line.

EXAMPLE 1. Find the distance from the line whose equation is $3x - 4y = 8$ to the point (7, 2).

Solution. To apply Formula X, first write the equation of the line in the slope-intercept form,

$$y = \tfrac{3}{4}x - 2$$

or

$$y - \tfrac{3}{4}x + 2 = 0.$$

Then the distance from the line to any point $P_1(x_1, y_1)$ is

$$d = \frac{y_1 - \frac{3}{4}x_1 + 2}{\sqrt{\frac{9}{16}+1}}.$$

Substituting 7 for x_1 and 2 for y_1, we have

$$d = \frac{2 - \frac{3}{4}\cdot 7 + 2}{\sqrt{\frac{9}{16}+1}}$$

$$= \frac{-\frac{5}{4}}{\frac{5}{4}} = -1.$$

To check the result draw the figure.

EXAMPLE 2. Find the distance between the parallel lines whose equations are $2x + 3y + 5 = 0$ and $2x + 3y - 7 = 0$.

Solution. The distance between two parallel lines is the distance from one of the lines to any point on the other. A point on the first line is $B(0, -\frac{5}{3})$.

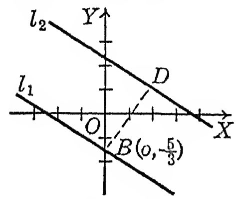


FIG. 29

To find the distance of B from the second line, we write its equation in the slope-intercept form,

$$y = -\tfrac{2}{3}x + \tfrac{7}{3}.$$

Then, by Formula X,

$$DB = \frac{-\frac{5}{3} - (-\frac{2}{3})(0) - (\frac{7}{3})}{\sqrt{\frac{4}{9} + 1}} = \frac{-12}{\sqrt{13}} = -\frac{12\sqrt{13}}{13} = -3.328.$$

Hence the distance between the lines is 3.328.

EXAMPLE 3. Find the area of the triangle whose vertices are $A(-1, 1)$, $B(6, -2)$, $C(3, 6)$.

Solution. The equation of AB is easily found to be

$$y = -\tfrac{3}{7}x + \tfrac{4}{7}.$$

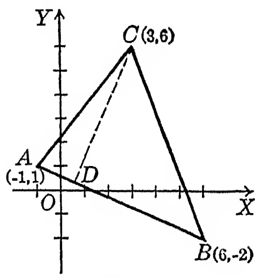


FIG. 30

The distance from AB to C is

$$d = \frac{6 - (-\frac{3}{7})(3) - \frac{4}{7}}{\sqrt{\frac{9}{49} + 1}} = \frac{47}{\sqrt{58}};$$

that is, the length of the altitude CD is $\dfrac{47}{\sqrt{58}}$.

The length of the base AB is given by the distance formula

$$AB = \sqrt{(-1 - 6)^2 + (1 + 2)^2} = \sqrt{58}.$$

Hence the area is $\dfrac{1}{2}AB \cdot CD = \dfrac{1}{2}\sqrt{58} \cdot \dfrac{47}{\sqrt{58}} = 23.5.$

The above method should be compared with that suggested in Problem 13, p. 36. The result obtained above can be roughly checked by counting the number of squares inside the triangle.

PROBLEMS

1. Find the distance from the given line to the given point. Draw the figure in each case.

a. $x + y - 5 = 0$, $(-2, 1)$.
b. $x - 2y + 4 = 0$, $(6, 2)$.
c. $12x + 5y - 2 = 0$, $(5, 7)$.
d. $3x - 4y - 5 = 0$, $(10, 4)$.
e. $x + 2y - 10 = 0$, $(7, 2)$.
f. $2x - 5y + 10 = 0$, $(6, 4)$.
g. $3x - 8y - 24 = 0$, $(3, -4)$.

2. Find the distance between each pair of parallel lines:

a. $3x - 4y + 7 = 0$, $3x - 4y - 8 = 0$.
b. $x - y + 12 = 0$, $x - y + 2 = 0$.
c. $5x + 12y - 30 = 0$, $5x + 12y + 20 = 0$.

3. Find the lengths of the three altitudes in each of the following triangles. Draw the figure in each case.

a. $(-5, 0), (-1, 6), (5, 2)$.
b. $(-2, 3), (-4, -5), (8, 6)$.
c. $(10, 0), (14, 6), (5, 10)$.
d. $(1, 7), (6, 3), (-4, -4)$.

4. Find the areas of the triangles in Problem 3.

5. Find the area of the quadrilateral whose vertices are $(-3, -4)$, $(6, -5)$, $(8, 4)$, $(-7, 10)$.

6. Find the equations of the bisectors of the angles between the lines whose equations are $y = x - 1$ and $y = 2x - 3$.

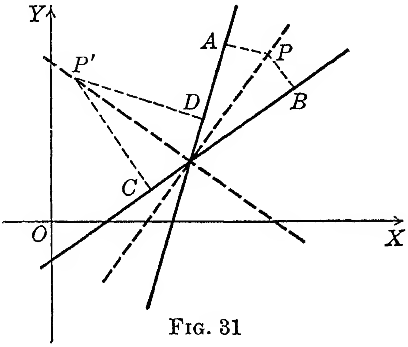


Fig. 31

Solution. We recall from plane geometry that any point on the bisector of an angle is equidistant from the sides. Thus, if allowance is made for difference in sign,

$$AP = -BP.$$

By Formula X this becomes

$$\frac{y-2x+3}{\sqrt{5}}=-\frac{y-x+1}{\sqrt{2}},$$

which, upon simplification, reduces to

$$(2\sqrt{2}+\sqrt{5})x-(\sqrt{2}+\sqrt{5})y-(3\sqrt{2}+\sqrt{5})=0.$$

Similarly, $$CP'=DP',$$

whence

$$(2\sqrt{2}-\sqrt{5})x+(\sqrt{5}-\sqrt{2})y+(\sqrt{5}-3\sqrt{2})=0.$$

7. Find the equations of the bisectors of the angles between the lines whose equations are $3x+4y-10=0$ and $3x-4y+2=0$.

8. Find the equations of the bisectors of the interior angles of the triangle defined by the equations $3x-4y-9=0$, $4x+3y-16=0$, and $12x-5y+24=0$.

9. Find the equations of the bisectors of the exterior angles of the triangle in Problem 8.

10. Find the equations of the bisectors of the interior angles of the triangle defined by the equations $x-10=0$, $y-3=0$, and $x-y=0$.

11. Find the equations of the bisectors of the exterior angles of the triangle in Problem 10.

12. Find the equations of the bisectors of the interior angles of the triangle whose vertices are $(10, 0)$, $(-2, 6)$, and $(-10, -10)$.

13. Find the equation of the locus of all points that are twice as far from the line whose equation is $3x-4y-3=0$ as from the x-axis.

14. Find the center and the radius of the circle inscribed in the triangle whose vertices are $(0, 10)$, $(12, 4)$, and $(-8, -6)$.

MISCELLANEOUS PROBLEMS

1. The vertices of a triangle are $A(8, 0)$, $B(6, 4)$, and $C(-1, 3)$. Find

a. The equation of AB.
b. The length of the altitude from C.
c. The area of the triangle.
d. The center of gravity.
e. The angles of the triangle.
f. The length of the median from C.
g. The equation of the perpendicular bisector of AB.
h. The center and the radius of the circumscribed circle.

2. Find the information called for in Problem 1 for the triangle whose vertices are $A(-5, 13)$, $B(7, 5)$, and $C(-10, -12)$.

3. Find the information called for in Problem 1 for the triangle whose vertices are $A(9, 0)$, $B(3, 3)$, and $C(-1, -5)$.

4. The vertices of a triangle are $A(9, 3)$, $B(3, 6)$, and $C(-1, -2)$. Find

a. The equations of the sides.
b. The lengths of the three altitudes.
c. The area.
d. The angles of the triangle.
e. The equations of the bisectors of the angles.
f. The center and the radius of the inscribed circle.
g. The equations of the perpendiculars from the center of the inscribed circle to the sides of the triangle.
h. The coördinates of the point of contact of the inscribed circle with each side of the triangle.

5. Find the information that is called for in Problem 4 for the triangle whose vertices are $A(5, 10)$, $B(11, -2)$, and $C(-5, -10)$.

6. Find the information called for in Problem 4 for the triangle whose vertices are $A(0, 0)$, $B(0, 12)$, and $C(4.5, 6)$.

7. The equations of the sides of a triangle are $4x-3y+9=0$, $3x+4y-17=0$, and $5x-12y-19=0$. Find

a. The angles of the triangle.

b. The equations of the bisectors of the angles.

c. The center and the radius of the inscribed circle.

d. The equations of the lines passing through the center of the inscribed circle and perpendicular to the sides of the triangle.

e. The coördinates of the point of contact of the inscribed circle with each side of the triangle.

8. Find the information called for in Problem 7 for the triangle whose sides have the equations $3x+2y-4=0$, $3x-2y+22=0$, and $2x-3y-7=0$.

9. Find the information called for in Problem 7 for the triangle whose sides have the equations $3x-4y+32=0$, $4x+3y+26=0$, and $x-4=0$.

10. The vertices of a parallelogram are $A(1,-2)$, $B(7,3)$, $C(6,9)$, $D(0,4)$. Find the altitude, regarding AB as the base. Find the altitude, regarding AD as the base. Check the results by finding the area in both cases.

11. The equations of the sides of a parallelogram are $5x+3y-30=0$, $5x+3y-1=0$, $3x-4y-18=0$, $3x-4y+11=0$. Find the point of intersection of the diagonals.

12. Find the area of the circumscribed circle of the triangle $2x+y-20=0$, $x+3y-10=0$, $x-7y+50=0$.

13. Find the area of the inscribed circle of the triangle $(2, 2)$, $(-4, 5)$, $(-8, -3)$.

14. The equations of the sides of a triangle are the following: $24x+7y-48=0$, $4x-3y-1=0$, $5x+12y-60=0$. On each side of the triangle find the point that is equidistant from the other two sides.

15. The vertex of an isosceles triangle is $(5, 7)$, and its base angles are $45°$. The equation of the base line is $x+2y-4=0$. Find the equations of the other two sides.

CHAPTER II

EQUATIONS OF CURVES

24. Equation of a Curve, or Locus. We have seen in the first chapter that the coördinates of every point on any given straight line satisfy a certain equation of the first degree and, conversely, that the locus of any given equation of the first degree is a certain straight line. In general it will be found that to every equation not of the first degree there corresponds a curve and that every curve has a definite equation. As a foundation for future work we have the following definitions, which must be thoroughly understood and memorized.

The equation of a curve, or locus, is the equation satisfied by the coördinates of every point on the curve and by those of no other point.

Conversely, *the graph, or locus, of an equation is a curve containing all points whose coördinates satisfy the equation, and no other points.*

These two definitions are really equivalent. From them we see that, in order to show that a particular point lies on a curve, we must substitute its coördinates in the equation of the curve and show that they verify the equation.

EXAMPLE 1. It will be shown later that the graph of $x^2 + y^2 - 20\,x = 0$ is a circle whose center is (10, 0) and whose radius is 10. In the figure the points (4, 8) and (17, $-$ 7) both

appear to lie on this circle. If we substitute 4 for x and 8 for y in the equation, we have

$$16 + 64 - 80 = 0;$$

if we substitute 17 for x and -7 for y, we have

$$289 + 49 - 340 = -2 \neq 0.$$

Hence the first point is on the circle, but the second is not.

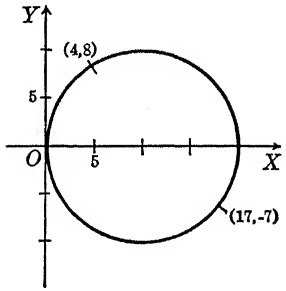


FIG. 32

EXAMPLE 2. Does the graph of $x^2 + 4xy + y^2 - 4x - y - 12 = 0$ pass through the origin? Where does it cut the x-axis?

Solution. Substituting $x = 0$ and $y = 0$ in the equation, we get $-12 = 0$, which is false. Hence the graph does not pass through the origin.

If it crosses the x-axis, $y = 0$ at the point of intersection. Substituting this in the equation, we get

$$x^2 - 4x - 12 = 0,$$

$$(x - 6)(x + 2) = 0;$$

whence $$x = 6 \text{ or } -2.$$

Therefore the curve cuts the x-axis at the points $(6, 0)$ and $(-2, 0)$.

25. General Problems related to a Curve and its Equation. As implied by the definitions in the preceding section, the study of curves by means of their equations involves two distinct, but closely related, problems. When the equation is known, the problem is to construct its graph and to study the properties of the curve by means of its equation. When a curve is known, or is defined by given properties, the problem is to find its equation.

The straight line has been treated in this manner in the first chapter, where it was shown that the graph of every

equation of the first degree is a straight line, which can be drawn by finding two points on the line, and that the equation of any straight line can be found by appropriate methods.

The construction of the graphs of equations of degree higher than the first is explained in §§ 26 and 27. The study of the properties of a curve by means of its equation is treated briefly in §§ 28 and 29 and more fully in Chapter VIII. The general method of finding the equation of a curve defined by given properties is illustrated in § 30.

26. Plotting Graphs of Equations. A few curves used in analysis are defined by geometric means, as is the circle in plane geometry, but usually a curve is defined by its equation. Since an equation in two unknowns is in general satisfied by an infinite number of pairs of values, we cannot plot all the points of a curve defined by an equation. But we can approximate the graph by finding a number of points whose coördinates satisfy the equation and by joining them with a smooth curve. This process is known as *plotting the graph of the equation.* In order to systematize the work it is customary to proceed in the following manner:

I. *Solve the equation for y in terms of x.* (Sometimes it is easier to solve for x in terms of y. If this is the case, interchange x and y in the following directions.)

II. (1) *Substitute convenient positive and negative values (usually integral) for x, and calculate the corresponding values of y.* Each pair of values of x and y satisfy the equation, and are therefore the coördinates of a point on the graph.

(2) *Arrange these pairs of values in a table, with the values of x increasing algebraically.*

III. *Plot each point, and join the points in the order of the table by a smooth curve.*

EXAMPLE. Plot the graph of the equation $x^2 - 2x - y = 0$.

x	y
-2	8
-1	3
0	0
1	-1
2	0
3	3
4	8

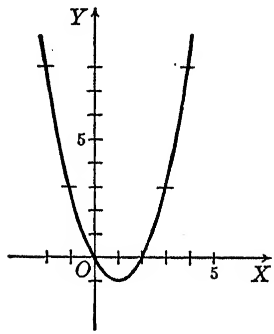


FIG. 33

Solving for y in terms of x, we have

$$y = x^2 - 2x.$$

Substituting $x = -2, -1, 0, 1, 2, 3, 4$, we get $y = 8, 3, 0, -1, 0, 3, 8$, respectively.

The points thus determined are tabulated at the left. If the table is extended by taking values of x greater than 4, it is apparent that the corresponding values of y will increase. The curve extends indefinitely to the right and upward. Similarly, by taking values of x less than -2, it can be seen that the curve extends indefinitely to the left and upward.

27. Further Remarks on Plotting. When the solution for y in terms of x involves a square root, both signs must be used before the radical and each value assigned to x will give two values of y and hence two points to be plotted.

The table of values should extend far enough to give a good idea of the shape of the graph.

The points where the curve crosses the axes, if there are any, should be found. This may be done as in the second example in § 24.

If the curve appears to change its direction abruptly at any point, look for an error in the table of values. If there is no such error, take intermediate fractional values of x (or y) and plot the corresponding points. This should also be done if the points plotted do not show clearly the form of the curve.

Applications of the above remarks are given in the solutions of the following illustrative examples:

EXAMPLE 1. Plot the graph of $4x^2 + y^2 = 24$.

x	y
-3	imag.
-2	± 2.8
-1	± 4.5
0	± 4.9
1	± 4.5
2	± 2.8
3	imag.

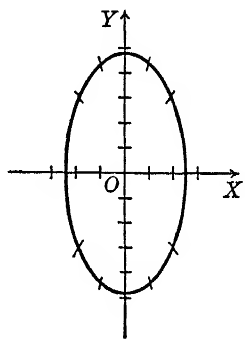


FIG. 34

Solving for y in terms of x,

$$y = \pm\sqrt{24 - 4x^2}$$
$$= \pm 2\sqrt{6 - x^2}.$$

A table of square roots or a slide rule will be found convenient in calculating the various values of this radical. As y is imaginary for $x > 3$, but real for $x = 2$, the curve must cross the x-axis between $x = 2$ and $x = 3$. These points are readily found by setting $y = 0$, whence

$$4x^2 = 24, \text{ or } x = \pm\sqrt{6} = \pm 2.4.^+$$

Adding the points (2.4, 0) and (— 2.4, 0) to the list, we can draw the entire curve.

EXAMPLE 2. Plot the graph of $y = x^3 - x$.

x	y
-3	-24
-2	-6
-1	0
0	0
1	0
2	6
3	24
$.5$	$-.375$
$-.5$	$+.375$

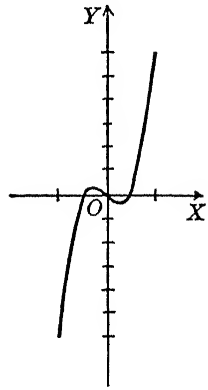


FIG. 35

There are no complications in finding the table of values. But when the points $(-1, 0)$, $(0, 0)$, and $(1, 0)$ are plotted, it is seen that the general rules for plotting appear to make the curve coincide with the x-axis in this region. By substituting $\pm \frac{1}{2}$ for x, we find two more points, which show that the curve rises above the x-axis between -1 and 0 and falls below it between 0 and 1.

EXAMPLE 3. Plot the graph of $xy = -16$.

Solving for y, we obtain $y = -\frac{16}{x}$. Two things should be noticed. First, as y decreases slowly after x becomes greater than 5, it is not advisable to space these values of x closely. Second, there is no point for which $x = 0$. For if 0 is substituted for x in the equation, we have $0 \cdot y = 0$, not -16. Taking fractional values of x, we have the supplementary table. This indicates that the curve recedes indefinitely along the y-axis as well as along the x-axis. Since y increases indefinitely as x approaches 0, it is customary to say that for $x = 0$, $y = \pm \infty$.

This abbreviation, however, should not be allowed to obscure the fact that for $x = 0$ there is *no* point on the curve.

x	y	x	y
1	-16	-1	16
2	-8	-2	8
3	-5.3	-3	5.3
4	-4	-4	4
5	-3.2	-5	3.2
10	-1.6	-10	1.6
16	-1	-16	1
32	-0.5	-32	0.5
$\frac{3}{4}$	-21.3	$-\frac{3}{4}$	21.3
$\frac{1}{2}$	-32.0	$-\frac{1}{2}$	32.0

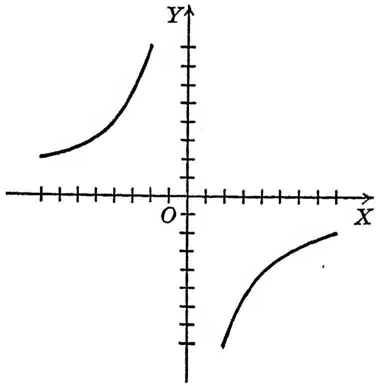


FIG. 36

PROBLEMS

1. Plot the graphs of the following equations:

a. $x = -3$.

b. $5x - 2y + 6 = 0$.

c. $y = x^2 - 6x + 8$.

d. $4y = 8 - 2x - x^2$.

e. $x^2 + y^2 = 32$.

f. $x^2 + y^2 = 36$.

g. $x^2 - y^2 = 4$.

h. $x^2 + 4y^2 = 36$.

i. $x^2 - 4y^2 = 16$.

j. $2x^2 + y^2 = 2$.

k. $y^2 = x^3$.

l. $y^3 = x^2$.

m. $y = x^3$.

n. $y^3 = x$.

o. $2xy = 9$.

p. $x^2y = 16$.

q. $xy^2 = 16$.

r. $x^2 + 4y = 0$.

s. $x^2 - y^2 + 16 = 0$.

t. $x^2y + 4y = 8$.

u. $x^2y + y = x$.

v. $x^2y - 4y = x$.

2. Find one of the curves in Problem 1 which passes through

a. $(0, 0)$. *b.* $(4, 0)$.

3. Plot the graphs of the following equations for the range of values indicated in each case:

Equation	*Range of Values*
a. $y^2 - 4y = x - 3,$	$y = -2$ to $y = 6.$
b. $x = y^2 + 5y + 4,$	$y = -7$ to $y = 2.$
c. $4x^2 - 9y^2 = 36,$	$x = -12$ to $x = 12.$
d. $x^2 - y^2 + 4x = 0,$	$x = -10$ to $x = 6.$
e. $y = x^3 - 4x,$	$x = -3$ to $x = 3.$
f. $2y = x^3 - 2x^2,$	$x = -2$ to $x = 4.$
g. $xy = 3y - 1,$	$x = -1$ to $x = 7.$
h. $xy - x^2 = 12,$	$x = -8$ to $x = 8.$
i. $5y = (x - 1)(x + 4)^2,$	$x = -6$ to $x = 3.$
j. $4y^2 = -x^3 + 4x^2,$	$x = -8$ to $x = 4.$
k. $y = x^3 - 9x - 5,$	$x = -3$ to $x = 4.$
l. $10y = x^4 - 9x^2,$	$x = -4$ to $x = 4.$
m. $y = (x - 1)(x + 1)(x - 2),$	$x = -3$ to $x = 4.$

4. Find one of the curves in Problem 3 which passes through

a. $(0, 0)$. *b.* $(4, 0)$.

5. Choose convenient values (other than 0) for the arbitrary constants in each of the following equations, and plot the respective graphs.

NOTE. If there are two arbitrary constants, do not use the same value for both.

a. $y = mx + b.$
b. $x^2 + y^2 = a^2.$
c. $b^2x^2 + a^2y^2 = a^2b^2$ (ellipse).
d. $b^2x^2 - a^2y^2 = a^2b^2$ (hyperbola).
e. $y^2 = 2px$ (parabola).
f. $x^2 = 2py$ (parabola).
g. $2xy = a^2$ (equilateral hyperbola).
h. $y = ax^3$ (cubical parabola).
i. $y^2 = ax^3$ (semicubical parabola).
j. $y = a + bx + cx^2$ (parabola).

6. Find the points where the graph of Problem 3, m, crosses the x-axis. Write the equation of a curve crossing the x-axis at (4, 0), (0, 0), and (– 2, 0) and draw its graph.

7. Plot the graph of $y = 4x^2 + C$ for two values of C, using the same coördinate axes. What is the effect upon the graph of changing the value of C?

28. Discussion of Equations. An effective study of the properties of most curves requires a knowledge of differentiation, and so this topic will be reserved for a later chapter. A cursory examination of the equation, however, will often yield considerable information about the curve and facilitate the labor of plotting. The properties most readily found by this means are the intercepts and the symmetry of the curve with respect to the coördinate axes.

Intercepts. The intercepts are the distances from the origin to the points of intersection of the curve and the coördinate axes. That is, the x-intercepts are the abscissas of those points on the curve for which $y = 0$; similarly, the y-intercepts are the ordinates of those points on the curve for which $x = 0$. To find the intercepts on the x-axis, substitute 0 for y and solve the resulting equation for x. To find the intercepts on the y-axis, substitute 0 for x and solve the resulting equation for y. (See § 24.)

Symmetry. If there are only even powers of x in the equation, it is evident that negative values substituted for x will give the same values of y as the corresponding positive values of x. Thus the points of the curve may be arranged in pairs, each pair symmetrical with respect to the y-axis, since the ordinate is the same and the abscissas differ only in sign (see also § 4). Such a curve is said to be symmetrical with respect to the y-axis. In this case the values of y when x is negative may be obtained at once from the values of y when x is positive.

Similarly, if there are only even powers of y in the equation, the curve is symmetrical with respect to the x-axis.

If the equation of a curve is unchanged when $-x$ is substituted for x and $-y$ for y, the points of the curve may be arranged in pairs, each pair symmetrical with respect to the origin. Such a curve is symmetrical with respect to the origin.

If a curve is symmetrical with respect to both axes, it is symmetrical also with respect to the origin; but a curve symmetrical with respect to the origin is not necessarily symmetrical with respect to both axes.

EXAMPLE 1. Investigate the intercepts and the symmetry of the graph of $4x^2 + y^2 = 24$.

Solution. When $y = 0$, $x = \pm\sqrt{6} = \pm 2.4$; when $x = 0$, $y = \pm\sqrt{24} = \pm 4.9$. Hence four points of the curve are $(\pm 2.4, 0)$ and $(0, \pm 4.9)$. As both x and y occur only with even exponents, the curve is symmetrical with respect to both axes and the origin, and a table of values for the first quadrant will suffice for the whole curve. This curve is plotted in the first illustrative example of § 27.

EXAMPLE 2. Investigate the intercepts and the symmetry of the graph of $y = x^3 - x$.

Solution. When $y = 0$, $x = 0$ or ± 1; when $x = 0$, $y = 0$. Hence the curve cuts the x-axis at three points: $(-1, 0)$, $(0, 0)$, and $(1, 0)$. The curve cuts the y-axis only at the origin. As both x and y occur with odd exponents, the curve is symmetrical with respect to neither axis. When $-x$ is substituted for x, and $-y$ for y, the equation becomes $-y = (-x)^3 - (-x)$. When the parentheses are removed, we have $-y = -x^3 + x$, which, by changing signs, is the same as the original equation. Hence the curve is symmetrical with respect to the origin. This curve is plotted in the second illustrative example of § 27.

Variation of the Coördinates. It is also frequently possible by inspection of the equation to determine whether y increases or decreases as x increases. Let us take as an illustration the equation discussed in Example 1. When we solve this for y in terms of x, in order to compute the table of values, we obtain $y = \pm 2\sqrt{6 - x^2}$. From this we see at once that as x increases numerically, y decreases numerically; also that x^2 can never be greater than 6, and hence that the largest possible value of $|x|$* is $\sqrt{6}$. Similarly, the largest possible value of $|y|$ is $2\sqrt{6}$.

29. Plotting by Factoring. It sometimes happens that when we transpose all the terms of an equation to one side, we can factor this expression. If so, the locus of the equation is the group of lines or curves obtained by setting each factor equal to zero and by plotting the loci of the equations thus obtained on the same coördinate axes.

EXAMPLE. Plot the locus of $x^2 - 3xy = -2y^2$.

Solution. Transposing $2y^2$ and factoring the expression, we have $(x - 2y)(x - y) = 0$. Setting each factor equal to zero, we have $x - 2y = 0$ and $x - y = 0$. The graphs of these equations are straight lines through the origin, with slopes $\frac{1}{2}$ and 1 respectively. Observe the curious result that, although the locus of an equation of degree higher than the first is never a straight line, it may sometimes be a pair of straight lines.

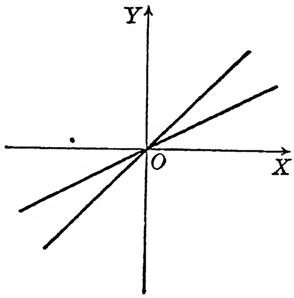


FIG. 37

The validity of the process just described follows at once from the definition of the locus of an equation. For if a point lies on either of the lines thus obtained, its coördinates

* $|x|$ means the value of x without regard to sign. Thus, $|-6| = |6| = +6$.

must satisfy one of the equations $x - 2y = 0$ or $x - y = 0$. But if either $x - 2y$ or $x - y$ is zero for a certain pair of values of x and y, their product, $x^2 - 3xy + 2y^2$, equals zero for the same values. Hence the point lies on the locus of the given equation. No other points have coördinates satisfying the given equation; for $x^2 - 3xy + 2y^2$ cannot equal zero for any values of x and y unless one of its factors equals zero, in which case the point would be on one of the lines.

PROBLEMS

1. Investigate the intercepts and the symmetry of the graph of each of the following equations:

a. $y = 8 - x^2$.

b. $x^2 + 4y^2 = 16$.

c. $2x^2 + y^2 = 1$.

d. $x^2 - 9y^2 = 36$.

e. $y^2 + 8 = x$.

f. $y^2 = \dfrac{x^3}{16}$.

g. $x^2y = 36$.

h. $x^2y + 4y = 12$.

i. $y = 9x - x^3$.

j. $y = x^4 - 9x^2$.

2. Each of the following questions refers to the corresponding equation in Problem 1. Answer the question, giving the reason for the statement made.

a. What is the largest value of y?

b. What are the largest numerical values of x and y?

c. What are the largest numerical values of x and y?

d. What is the smallest numerical value of x?

e. What is the smallest value of x?

f. In which quadrants does this curve lie?

g. How does y change as x increases?

h. What is the largest value of y? Is y ever negative?

i. For what values of x is y positive?

j. For what values of x is y positive?

3. Using the results obtained in Problem 1, and such other information as the equation may yield, endeavor to sketch each of the curves without obtaining a table of values.

4. Which of the curves in Problem 1, p. 55, have symmetry

a. With respect to the x-axis?
b. With respect to the y-axis?
c. With respect to both axes?
d. With respect to the origin only?

5. In the equation $4y = x^2 - 10x + 24$ substitute $-x$ for x and plot the graphs of both the new equation and the original equation on the same axes. In what way are the curves related?

6. Plot on the same axes the loci of $2y = x^2 - 16$ and $2y = 16 - x^2$. How are these loci related?

7. Plot on the same axes the loci of $y = x^3$ and $x = y^3$. How are these loci related?

8. For what value of p will the graph of $y^2 = 2px$ pass through the point (16, 4)? Plot the graph.

9. For what values of the constants a and b will the graph of $b^2x^2 + a^2y^2 = a^2b^2$ pass through (1, 3) and (4, -2)? Plot the graph.

10. For what values of a, b, and c will the graph of $y = a + bx + cx^2$ pass through the points (-1, 1), (1, 7), and (3, 5)? Plot the graph.

11. What is the locus of

a. $x^2 + y^2 = 0$?
b. $(x - 2)^2 + (y + 3)^2 = 0$?

12. Show that $x^2 + y^2 + 6 = 0$ has no real locus.

13. Plot by factoring the loci of

a. $xy = 0$.
b. $y^2 + 5y - 6 = 0$.
c. $4x^2 - y^2 = 0$.
d. $4x^2 - y^2 - 8x + 4y = 0$.
e. $x^2 - y^2 + 10x - 2y + 24 = 0$.
f. $(3x + 2y)^2 + (3x + 2y) = 2$.
g. $2x^2 - 3xy + 4x = 6y$.
h. $(x^2 + y^2 - 25)(x^2 + y^2 - 16) = 0$

14. What is the locus of $(x-k)(y-k)=0$ if k is allowed to take on all positive and negative integral values?

15. Write a single equation whose locus is the pair of lines

a. Whose equations are $x-y=0$ and $x=-y$.

b. Whose equations are $x-2y=6$ and $2x+y=4$.

c. Which are parallel to the y-axis and which have the x-intercepts 2 and -2.

d. Which pass through the point (1, 2) and have slopes -1 and $+1$.

16. Find the equations of the bisectors of the angles between the lines defined by $2x^2-5xy-3y^2+12x+6y=0$.

30. Derivation of Equations. In order to find the equation of a given curve or of the locus of a point satisfying certain given conditions, it is customary to draw a figure showing the data of the problem, to take a *general* point $P(x, y)$ satisfying the given conditions, and to express these conditions in the form of an equation containing x and y and no other variables. The following examples illustrate the method:

EXAMPLE 1. Find the equation of the circle with the center at (10, 0) and with the radius 10.

Solution. Let $P(x, y)$ be any point on the circle of radius 10 with center at $C(10, 0)$. Then $CP=10$, by the definition of a circle.

But $CP=\sqrt{(x-10)^2+y^2}$ by the distance formula.

Hence $\sqrt{(x-10)^2+y^2}=10$.

After squaring and simplifying we obtain

$$x^2+y^2-20x=0,$$

which is the required equation of the circle.

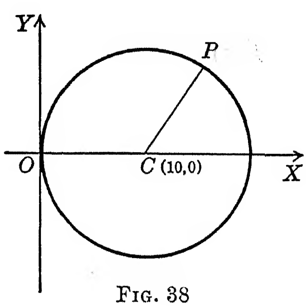


FIG. 38

EXAMPLE 2. Find the equation of the locus of a point equidistant from the line $y = -8$ and the point $(6, 2)$.

Solution. In the figure the given point is A and the given line is BC. Let $P(x, y)$ be a general point which seems to fulfill the condition of the problem. Since P is to be equidistant from A and BC, we draw AP and the line EP perpendicular to BC. By the statement of the problem these lines are equal; that is,

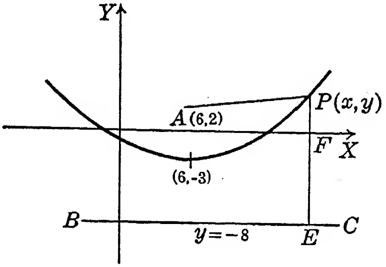


FIG. 39

$$AP = EP.$$

We can replace AP by $\sqrt{(x-6)^2 + (y-2)^2}$, using the distance formula.

Moreover, $$EP = EF + FP = 8 + y.$$

Therefore $$\sqrt{(x-6)^2 + (y-2)^2} = 8 + y.$$

Squaring and simplifying, we have

$$20\,y = x^2 - 12\,x - 24.$$

The graph should now be plotted on the original figure, so as to show its shape and relation to the given point and line.

EXAMPLE 3. The ends of a line of variable length are on two fixed perpendicular lines. Find the locus of the mid-point of this line if the area of the triangle thus formed is 24.

Solution. The perpendicular lines should be taken as the coördinate axes. Let AB be one of the positions of the variable line and take $P(x, y)$ as its mid-point. The conditions of the problem are

$$AP = PB \quad \text{and} \quad \text{area of } AOB = 24.$$

The latter equation gives at once

$$\frac{OA \cdot OB}{2} = 24.$$

We must now express OA and OB in terms of x and y. Draw the coördinates of P, $EP = x$ and $FP = y$. The figure suggests the theorem of plane geometry that a straight line parallel to the base of a triangle and bisecting one side bisects the other.

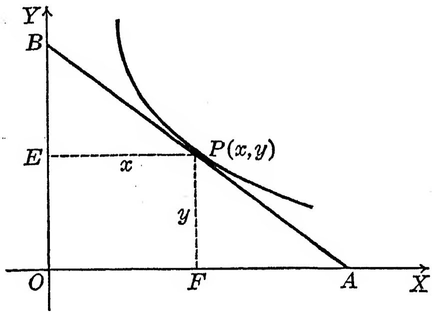


FIG. 40

Hence

$$OA = 2\ OF = 2\ x$$

and $OB = 2\ y$.

Substituting above, we get

$$2\ xy = 24,$$

or $$xy = 12,$$

which is the equation of the locus. The curve can now be drawn in the usual manner.

NOTE. In none of the above examples is it shown that the equation obtained is satisfied by the coördinates of no point not on the locus, or, what amounts to the same thing, that every point whose coördinates satisfy the equation lies on the locus. This can be done by simply retracing the steps in the derivation of the equation in reverse order. This, however, is seldom necessary.

PROBLEMS

1. Find the equation of the locus of points equidistant from

a. $(-3, 6)$ and $(10, -2)$. *c.* $(9, 6)$ and $(15, -6)$.
b. $(1, 4)$ and $(4, 1)$. *d.* $(0, 6)$ and $(4, 2)$.

2. Find the equation of the locus of points equidistant from

a. The point $(-6, 0)$ and the line $x = 16$.
b. The point $(0, 8)$ and the line $y = 5$.
c. The point $(1, 4)$ and the line $y = 0$.
d. The point $(8, 6)$ and the line $x = 12$.

3. Find the locus in each case of Problem 2 if the distance from the point is always two units greater than the distance from the line.

4. The base of an isosceles triangle is OB, where $O = (0, 0)$ and B is a variable point on the x-axis. Find the locus of the third vertex P if the area of OBP is 16.

5. Find the equation of the circle whose center and radius are as follows:

a. Center $(4, -5)$, radius $= 10$.
b. Center $(-6, 0)$, radius $= 6$.
c. Center $(0, -5)$, radius $= 5$.
d. Center $(0, 0)$, radius $= r$.
e. Center (h, k), radius $= r$.

6. Find the locus of a point if its distance from the point $(-4, 0)$ is always twice its distance from the point $(2, 0)$.

7. Find the locus of a point if it is always three times as far from the origin as from the point $(4, 4)$.

8. The base of a triangle is AB, where A is $(-3, 0)$ and B is $(3, 0)$. Find the locus of the vertex P if the slope of BP is 1 unit greater than the slope of AP.

9. Find the locus of a point whose distance from the origin is a mean proportional between its distances from the points $(2, 0)$ and $(-2, 0)$.

10. The ends of the hypotenuse of a right triangle are $(2, 2)$ and $(4, -2)$. Find the equation of the locus of the vertex of the right angle.

11. The base of a triangle is AB, where A is $(-5, 0)$ and B is $(5, 0)$. Find the locus of the third vertex P if

a. $(AP)^2 - (BP)^2$ is a constant, k.
b. $(AP)^2 + (BP)^2$ is a constant, k.
c. The slope of BP is k times the slope of AP.
d. The sum of the slopes is a constant, k.

EXAMPLE. What are the points of intersection of the curves whose equations are $x^2 + y^2 = 36$ and $x^2 = 5y$?

Solution. The successive steps in solving the equations simultaneously are

$$5y + y^2 = 36;$$
$$y^2 + 5y - 36 = 0;$$
$$(y+9)(y-4) = 0;$$
$$y = -9 \text{ or } 4.$$

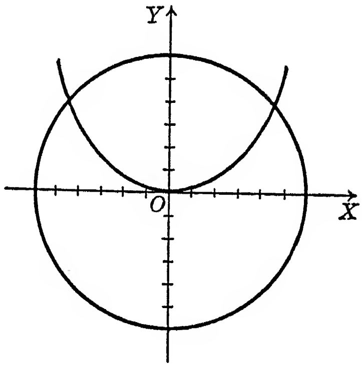


FIG. 41

Substituting these values in the equation $x^2 = 5y$, we have for $y = 4$,

$$x = \pm\sqrt{20} = \pm 4.47;$$

for $y = -9$, $x = \sqrt{-45}$.

Thus there are two common points (4.47, 4) and (− 4.47, 4). The graphs furnish a convenient check.

PROBLEMS

Find the points common to the loci of the following equations and plot the loci. If the solutions are fractional or irrational, express the results to the nearest tenth.

1. $4x - y = 4$, $y^2 - 8x = 0$.

2. $x^2 + y^2 = 41$, $4x - 5y = 41$.

3. $4x^2 + 9y^2 = 36$, $4x^2 + 3y = 30$.

4. $x^2 + y = 7$, $y^2 - 81 = 0$.

5. $x^2 + y^2 = 64$, $x + 2y = 2$.

6. $y = x^2 - 6x + 8$, $y = 3x - 9$.

7. $x^2 + y^2 = 16$, $xy = 7$.

8. $x^2 - y^2 = 64$, $x - 2y = 0$.

9. $x^2 - y^2 = 64$, $2x - y = 0$.

10. $x^2 = y$, $y^2 = 2x$.

11. $y = \dfrac{8}{4 + x^2}$, $x^2 = 4y$.

12. $y = x^3$, $y = x^2 + 2x$.

13. $x^2y + y - x = 0$, $4y = x$.

14. $3y = x^3 - 6x$, $x + y = 0$.

15. Find the points on the line whose equation is $x - 7y = 34$ which are 5 units distant from the point $(2, -1)$.

HINT. First find the equation of the circle whose radius is 5 and whose center is $(2, -1)$.

16. Find the length of the chord of the circle $x^2 + y^2 = 64$ cut off on the line $x + y = 2$.

17. Find the distance, measured along the line $y = 2x$, from the origin to the ellipse $4x^2 + 9y^2 = 36$.

18. Find the coördinates of the points 4 units distant from $(1, 5)$ and 5 units distant from $(-3, 4)$.

19. For what values of m will the line $y = mx$ fail to meet the curve $y = x^2 + 5$?

MISCELLANEOUS PROBLEMS

1. The equation of an ellipse 160 yards long and 80 yards wide is

$$\frac{x^2}{6400} + \frac{y^2}{1600} = 1.$$

Plot this curve, taking values of x at 20-yard intervals, and find by a proper drawing whether or not this field will contain a rectangle 100 yards long and 50 yards wide.

2. For what values of m will the line $y = mx$ cut the locus of the equation

$$y = \frac{x}{1 + x^2}$$

in three points? Plot the curve and the line for the case that $m = \frac{1}{5}$.

3. Plot carefully on the same axes the graphs of $y^2 = 8x$ and $y = 2 + 6x - x^2$. From the graphs estimate, correct to tenths, the coördinates of the points of intersection. What difficulty is met with if it is attempted to find these by solving the equations simultaneously?

4. Treat in the same way as in Problem 3 the equations

$$x - y = 5 \quad \text{and} \quad y = \frac{x^3}{8} - \frac{3x^2}{4}.$$

5. The radii of two circles are 6 and 8, and their centers are 10 units apart. Take the centers in a convenient position, find the equations of the circles and their points of intersection, and get the length of the common chord.

6. Prove that the equation of a circle whose center is (h, k) and whose radius is r is $(x-h)^2+(y-k)^2=r^2$.

7. Plot the system of curves whose equation is $x^2+y^2=r^2$.

8. A point moves in such a way that the sum of its distances from the two points $(3, 0)$ and $(-3, 0)$ is always 10. Find the equation of its locus and plot it.

9. A point moves in such a way that the difference of its distances from the points $(5, 0)$ and $(-5, 0)$ is always 8. Find the equation of its locus and plot it.

10. Plot on the same axes three curves of the system whose equation is $y^2=kx$.

11. The base of a triangle is AB, where $A=(5, 0)$ and $B=(-5, 0)$. Find the equation of the locus of the vertex if the vertical angle is 45°.

HINT. Use the formula for $\tan\theta$, § 14.

12. Plot the graph of $y^2=ax^2-x^3$.

13. Plot on a large scale $y=x^3-x^2-4x+3$ for x ranging from -2 to 3, and estimate from the graph the values of x which make y zero.

14. Plot carefully the graph of $y=x^2$ and draw the straight line $y=2x+5$. Estimate the abscissas of the points of intersection and show that these are the roots of the equation $x^2-2x-5=0$. Hence state a method of solving any quadratic graphically.

15. Using the method developed in Problem 14, solve graphically the quadratics

a. $x^2-12x-43=0$.

b. $2x^2-6x+1=0$.

16. If the parabola and the straight line in Problem 14 did not meet, what could be said of the roots of the quadratic?

CHAPTER III

SLOPE AND DERIVATIVE

32. Functions. Let us consider the coördinates of a point moving along a curve, as in the adjoining figure. As the point P moves, the values of the coördinates x and y are *variable* and are mutually dependent. For each value of x there are one or more definite values of y, and likewise for each value of y there are one or more definite values of x. In this case we say that y is a function of x and vice versa. More precisely, we have the definition:

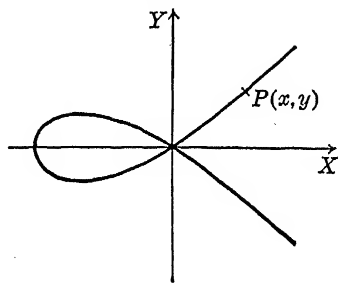


FIG. 42

If one of two variables has one or more definite values corresponding to each value given to the other variable, the first variable is a function of the second.

Although for the present we shall confine our discussion chiefly to the case in which the variables are the coördinates of a point on a curve, there are many other instances of functional dependence. The volume of an expanding sphere is a function of the radius. The velocity of a falling body is a function of the distance through which it has fallen. The premium paid on a $1000 life-insurance policy is a function of the age of the insured.

When the conditions of the definition are realized, the first variable is called the *dependent variable* or the *function*; and the second variable is called the *independent variable* or the *argument*. If, in the case of a moving point on a curve, we are considering the ordinate as depending upon the abscissa, y is the dependent variable and x is the independent variable.

33. Functional Notation. To indicate that y is a function of x, we write $y=f(x)$. This is read as "y equals f of x," or "y equals a function of x," and must not be confused with $y=f\cdot x$. Similarly, $y=f(x, t)$ denotes that y is a function of the two variables x and t.

Furthermore, the notation $y=f(x)$ may be used to represent the particular way in which y is related to x. For example, if $y=x^3-x^2+5x-6$, we may represent this equation by $y=f(x)$, in which case $f(x)$ stands for the expression x^3-x^2+5x-6 throughout the discussion. In short, the symbol $f(x)$ may be used to represent any mathematical expression involving x as a variable, or any quantity which is a function of x. To denote several different functions of x, we use various letters, as $f(x)$, $g(x)$, $\phi(x)$, etc.

If $f(x)$ represents a certain expression, $f(0)$ is used to denote the value of that expression when 0 is substituted for x; $f(-6)$ is used to denote the value of the expression when -6 is substituted for x; etc.

For example, let

$$f(x)=1+2x-x^2.$$

Then

$$\begin{aligned}
f(0)&=1+0-0=1,\\
f(3)&=1+2\cdot 3-3^2=-2,\\
f(-6)&=1+2(-6)-(-6)^2=-47,\\
f(t)&=1+2t-t^2,\\
f(2x)&=1+2(2x)-(2x)^2=1+4x-4x^2, \text{ etc.}
\end{aligned}$$

To write an equation containing x and y in the form $y=f(x)$ means simply to solve the equation for y in terms of x. If, for example, we wish to write $x^2+y=6x$ in the form $y=f(x)$, we obtain $y=6x-x^2$. Similarly, $x=g(y)=3\pm\sqrt{9-y}$.

34. Increments. *If a variable changes from one numerical value to another, the difference found by subtracting the first value from the second is called the increment of the variable.* Thus, if x is first taken as 4 and then as 7, the increment of x is 3. The term *increment* is used even when the variable decreases; thus, if x is first 4 and then 3, we say that the increment of x is $3-4=-1$.

It is customary to denote an increment by the letter Δ (*delta*) prefixed to the letter designating the variable; thus, Δx denotes an increment of x, Δy an increment of y, $\Delta f(x)$ an increment of $f(x)$, etc. It should be noted that the symbol Δx is a composite symbol, not to be treated as Δ times x. For example, $x\cdot\Delta x$ means x times Δx and may not be written $\Delta\cdot x^2$; and $\Delta x\Delta y$ cannot be combined as $\Delta^2 xy$. Furthermore, if Δx is negative, it is not written $-\Delta x$, but is treated like any other algebraic quantity.

If y is a function of x, and x takes on an increment Δx, then y will take on a *corresponding* increment Δy. In order to calculate the increment Δy we proceed as follows:

Suppose that $y=f(x)$. If x is given a new value, $x+\Delta x$, y will acquire a new value, $y+\Delta y=f(x+\Delta x)$. The increment Δy is obtained by subtracting the old value from the new; hence

$$\Delta y=f(x+\Delta x)-f(x).$$

EXAMPLE 1. If $y=x^2-6x$, compute Δy when $x=2$ and $\Delta x=0.3$.

Solution. When $x=2$,

$$y=f(x)=2^2-6\cdot 2=4-12=-8.$$

Since $$\Delta x = 0.3,$$
$$x + \Delta x = 2.3$$
and $$y + \Delta y = f(x + \Delta x) = (2.3)^2 - 6(2.3) = -8.51.$$

Subtracting y from $y + \Delta y$, we have
$$\Delta y = -8.51 - (-8) = -0.51.$$

EXAMPLE 2. If $y = x^2 - 6x$, find Δy for any x and Δx.

Solution. When $x = x$,
$$y = f(x) = x^2 - 6x. \tag{1}$$

When $x = x + \Delta x$,
$$\begin{aligned} y + \Delta y = f(x + \Delta x) &= (x + \Delta x)^2 - 6(x + \Delta x) \\ &= x^2 + 2x\Delta x + (\Delta x)^2 - 6x - 6\Delta x. \end{aligned} \tag{2}$$

Subtracting equation (1) from equation (2), we have
$$\Delta y = 2x\Delta x + (\Delta x)^2 - 6\Delta x,$$
which is the result desired.

The value of Δy for particular values of x and Δx may be obtained by substitution in the result just obtained. For example, if we set $x = 2$ and $\Delta x = 0.3$, we obtain
$$\begin{aligned} \Delta y &= 2(2)(0.3) + (0.3)^2 - 6(0.3) \\ &= -0.51, \end{aligned}$$
as in Example 1.

NOTE. In these examples and the preceding paragraphs we have merely clothed a familiar problem in new language. To speak of the increment Δy corresponding to the increment Δx is another way of speaking of the change in y due to a change in x. Thus Example 1 can be worded in more familiar language as follows: If $y = x^2 - 6x$, how much will y change when x changes from 2 to 2.3?

By the introduction of the symbols Δx and Δy we are able to deal more precisely with changes in functions, as will be shown later. We have already seen in Example 2 how we can gèneralize the simple problem of Example 1 by the aid of these symbols.

PROBLEMS

1. Write in the forms $y = f(x)$ and $x = g(y)$ the following equations:

a. $x^2 + 2y^2 = 4$.

b. $x^2 - y^2 = 16$.

c. $y = 3x - x^2$.

d. $y^2 = x^4 - 2x^2$.

2. If $y = f(x)$ is the equation of a curve, show that the ordinate of the point whose abscissa is -2 is $f(-2)$.

3. If $y = g(x)$ is the equation of a curve, show that the y-intercepts of the curve are the values of $g(0)$.

4. If $y = f(x)$ is the equation of a certain curve, what is the graph of the equation $y = 2 + f(x)$?

5. Find

a. $f(0)$ and $f(-1)$ if $f(x) = \sqrt{1 + x^2} - x$.

b. $f(\sqrt{2})$ and $f(-6)$ if $f(t) = 6 - 3t^2$.

c. $g(0)$ and $g\left(\frac{\pi}{2}\right)$ if $g(\theta) = \sin 2\theta$.

d. $\phi(a)$ and $\phi(-x)$ if $\phi(x) = \sqrt{a^2 - x^2} - x$.

6. If $f(x) = \sin x$, prove that

a. $f(\pi - x) = f(x)$.

b. $f(2x) = 2f(x)\sqrt{1 - [f(x)]^2}$.

7. If $\phi(x) = a^x$, prove that $\phi(x) \cdot \phi(y) = \phi(x + y)$.

8. Find the increments of the following functions for the indicated values of the independent variable and of the increment of the independent variable:

a. $y = 2 + 3x - x^2$; $x = 2$, $\Delta x = 0.1$.

b. $y = \sqrt{25 - x^2}$; $x = 3$, $\Delta x = 0.2$.

c. $p = \dfrac{36}{v}$; $v = 5$, $\Delta v = 0.4$.

d. $f(t) = 1 - 2t^2$; $t = -4$, $\Delta t = 0.3$.

e. $y = x + \sqrt{x}$; $x = 6$, $\Delta x = 0.25$.

f. $z = 4u - u^2$; $u = 2$, $\Delta u = -0.2$.

9. For the following equations find Δy in terms of x and Δx:

a. $y = 4x^2 - 5x$. *c.* $y = 16 - x^2$. *e.* $y = x^2 + 5x - 6$.
b. $y = x^3$. *d.* $xy = 36$. *f.* $y = ax^2 + bx + c$.

35. Direction along a Curve. The most striking characteristic of a curve is, of course, that it is not straight; in other words, a point moving along a curve is constantly changing its direction of motion. The direction of motion at a point is considered to be along the tangent to the curve at that point. This axiom is based upon experience, for bodies moving in a curve always tend to move along the tangent, or, as we sometimes say, "to fly off at a tangent," unless kept in the curved path by some force.

In order to make our work general, we must give a definition of *tangent* which is different from that used in elementary plane geometry. The definition which we shall use is as follows:

The tangent to a curve at the point P is the limiting position of a secant through P and a second point Q on the curve, when Q approaches P as a limit.

In the figure, Q, Q', Q'', Q''' are several successive positions of point Q as it approaches P. Evidently the secant PQ rotates about P and approaches a limiting position PT where it just touches the curve at P.

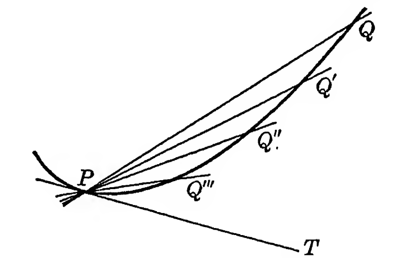


FIG. 43

The definition of *tangent* used in elementary geometry — namely, as "a line touching the circle at one and only one point" — must be modified for other curves, as the following figures show.

In figure (*a*) the line *TPM* is tangent to the curve at the point *P*. The tangent "touches" the curve at *P*, but also "cuts" it again at *M*.

In figure (*b*) the line *TP* is tangent to the curve at *P*. The tangent "cuts" the curve at the point of tangency instead of "touching" it.

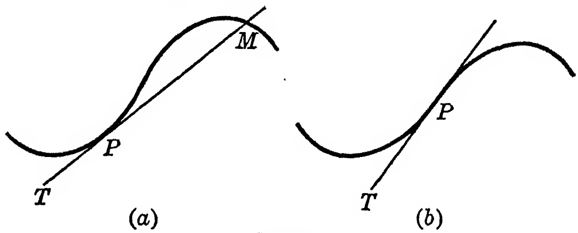


FIG. 44

In the case of the circle both definitions give the same line, as may be seen intuitively, and as will be proved later (§ 94).

36. Slope of a Curve. The slope of a curve at a point is defined as the slope of the tangent at the point. Similarly, the inclination of a curve is the inclination of the tangent. The slope of a curve, unlike that of a straight line, changes from point to point.

We can approximate the slope of a curve at any point P by taking a point Q on the curve near P and getting the slope of the secant PQ. Let us take as an example the curve whose equation is $y=f(x)=4x-x^2$ and try to find its slope at the point $P(x_1, y_1)=(1, 3)$. Taking $Q(x_2, y_2)$ as the point whose abscissa is 1.1, we find that

$$y_2=f(1.1)=4.4-1.21=3.19;$$

that is, Q is (1.1, 3.19). By the slope formula, the slope of PQ is

$$\frac{3.19-3}{1.1-1}=\frac{.19}{.1}=1.9.$$

The figure shows that the slope of the tangent cannot be very much different from this number.

If $x_2 = 1.01$, $y_2 = 3.0199$. For this position of Q the slope of the secant is 1.99. If Q is taken as (.99, 2.9799), we find that the slope of the secant is 2.01. It appears probable, then, that as Q approaches P the slope of PQ approaches 2, which we assume is the slope of the tangent PT. In other words, at P the point tracing the curve is rising twice as fast as it is moving to the right.

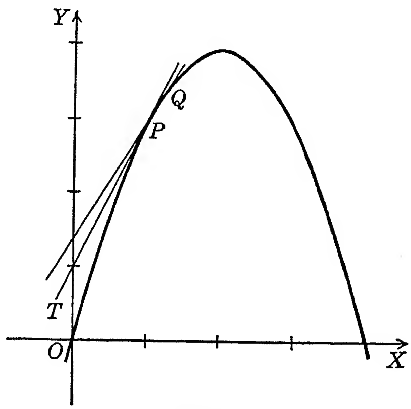


FIG. 45

The method just employed is open to two objections: first, it is tedious; secondly, we have not really proved that the slope at (1, 3) is 2 (it may be 1.99998). Both these objections are met by the use of increments, as will be shown in the following sections.

37. Exact Slope of a Curve at a Given Point. With the aid of increments we are able to find the exact slope of a curve at any given point. To make the problem definite, let the equation of the curve be $y = x^2$ and let the point be $P(1, 1)$. If Q is another point on the curve, its abscissa is $1 + \Delta x$, where Δx is the horizontal distance from P to Q, and its ordinate is $1 + \Delta y$, where Δy is the vertical distance from P to Q. The slope of the secant PQ is easily seen to be $\frac{\Delta y}{\Delta x}$, either by use of the slope formula or from the figure.

Now the tangent at P is defined as the limiting position of the secant PQ as Q moves along the curve and approaches P as a limit. But as $Q \to P$,* $\Delta x \to 0$. (Does $\Delta y \to 0$?) Hence,

$$\text{Slope of tangent} = \text{limit of slope of secant}$$
$$= \text{limit of } \frac{\Delta y}{\Delta x} \text{ as } \Delta x \to 0.$$

Hence, *if we can find an expression for* $\frac{\Delta y}{\Delta x}$ *and calculate its limit as* $\Delta x \to 0$, *we can find the slope of the tangent to the curve at the given point.*

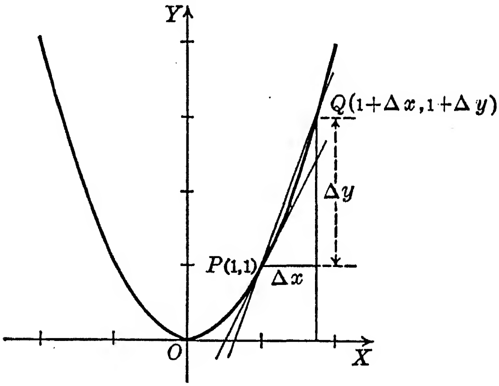


FIG. 46

In order to do this we make use of the fact that the coördinates of Q must satisfy the equation of the curve.

Then $$1+\Delta y = (1+\Delta x)^2,$$
whence $$\Delta y = 2\,\Delta x + (\Delta x)^2.$$

Dividing by Δx, $$\frac{\Delta y}{\Delta x} = 2 + \Delta x,$$
which is the slope of the secant PQ.

* The symbols $\to$ and $\doteq$ are used for the word *approaches*. The former is preferable, as it is not easily confused with the sign of equality.

From this equation it is apparent that $\frac{\Delta y}{\Delta x} \rightarrow 2$ when $\Delta x \rightarrow 0$. Hence the slope of the tangent to the curve $y = x^2$ at the point (1, 1) is 2.

If we consider the above work as a whole, we see that there are two principles involved, and they should be memorized. They are

1. The slope of any secant through P is $\frac{\Delta y}{\Delta x}$.
2. The slope of the tangent at P is the limit of $\frac{\Delta y}{\Delta x}$ as $\Delta x \rightarrow 0$.

In order to show the essentials of the work in condensed form, another example will be given.

EXAMPLE. Find the slope and inclination of the tangent to the curve whose equation is $y = 4x - x^2$ at the point (1, 3).

Solution. Let m denote the required slope and let Q be a point on the curve, with coördinates $(1 + \Delta x, 3 + \Delta y)$.

Then

$$\begin{aligned} 3 + \Delta y &= 4(1 + \Delta x) \\ &\quad - (1 + \Delta x)^2 \\ &= 4 + 4\Delta x - 1 \\ &\quad - 2\Delta x - (\Delta x)^2 \\ &= 3 + 2\Delta x - (\Delta x)^2. \end{aligned}$$

Hence

$$\Delta y = 2\Delta x - (\Delta x)^2,$$

and $$\frac{\Delta y}{\Delta x} = 2 - \Delta x.$$

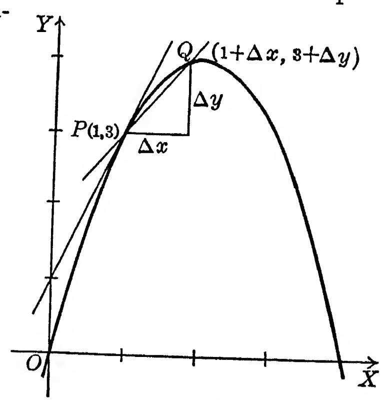


FIG. 47

Letting Δx approach 0, we have

$$m = 2.$$

(Compare this result with § 36.)

The inclination is found from the tables to be 63.43° (63° 26′).

In drawing a tangent always use the method of § 12. That is, since the slope is 2, the tangent rises at the rate of 2 vertical units per horizontal unit, or 10 vertical units for every 5 horizontal units, etc.

38. Slope of a Curve at any Point. Let us now consider the problem of finding the values of the slope at several points on a given curve. The method explained in § 37 permits us to find the slope for each point, but we must work the problem as many times as we have points. This may be avoided by finding a general formula for the slope at *any* point. How this is done is shown in the following illustrative examples, which should be compared with that of § 37.

EXAMPLE 1. Find the slope of the curve whose equation is $y = 4x - x^2$: (a) at any point; (b) at each of the points whose abscissas are 1, 2, and 3.

Solution. Let $P(x, y)$ be any *fixed* point on the curve and let $Q(x + \Delta x, y + \Delta y)$ be a point on the curve near P.

Then, from the equation,

$$y = 4x - x^2$$

and

$$y + \Delta y = 4(x + \Delta x) - (x + \Delta x)^2$$
$$= 4x + 4\Delta x - x^2 - 2x\Delta x - (\Delta x)^2.$$

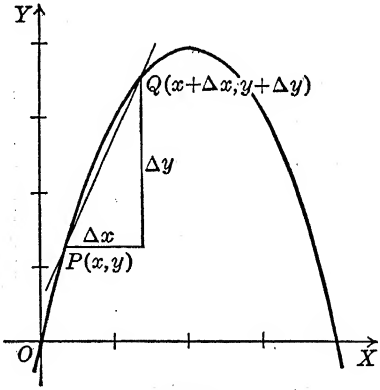


FIG. 48

Subtracting, $\Delta y = 4\Delta x - 2x\Delta x - (\Delta x)^2.$

Dividing by Δx, we have

$$\frac{\Delta y}{\Delta x} = 4 - 2x - \Delta x,$$

which is the slope of the secant PQ.

But, by § 37, the slope of the tangent is the limit of $\frac{\Delta y}{\Delta x}$ as Δx approaches 0. In this case the limit is obviously $4 - 2x$.

Hence, if m represents the slope at any point $P(x, y)$,

$$m = 4 - 2x.$$

To find the slopes at the particular points mentioned in (*b*), we simply substitute the values of x there given, obtaining the following results:

$$\text{For } x = 1, \quad m = 4 - 2 = 2.$$
$$\text{For } x = 2, \quad m = 4 - 4 = 0.$$
$$\text{For } x = 3, \quad m = 4 - 6 = -2.$$

These results may be checked by drawing the tangents at these points in the above figure.

EXAMPLE 2. Find the slope of the curve whose equation is $xy = 24$: (*a*) at any point; (*b*) at each of the points whose abscissas are 1, 2, 3, 4, 6, 8, 12.

Solution. Proceeding as in the previous example,

$$y = \frac{24}{x},$$

$$y + \Delta y = \frac{24}{x + \Delta x}.$$

Subtracting,

$$\Delta y = \frac{24}{x + \Delta x} - \frac{24}{x}$$
$$= \frac{-24\Delta x}{x(x + \Delta x)}.$$

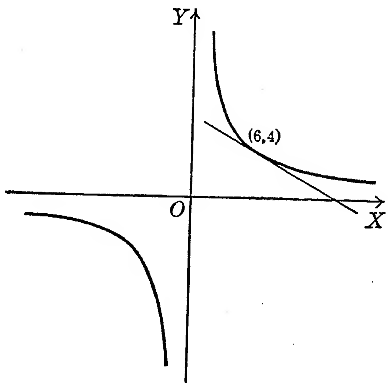


FIG. 49

Dividing by Δx, $\qquad \frac{\Delta y}{\Delta x} = \frac{-24}{x(x + \Delta x)}.$

Letting $\Delta x \rightarrow 0$, $\qquad m = \frac{-24}{x^2},$

which is the slope of the tangent at any point.

The values of the slope at the points specified in (*b*) may be obtained by simply substituting in the expression for m. They are tabulated below:

x	1	2	3	4	6	8	12
m	-24	-6	$-\frac{8}{3}$	$-\frac{3}{2}$	$-\frac{2}{3}$	$-\frac{3}{8}$	$-\frac{1}{6}$

The figure shows the tangent at (6, 4).

PROBLEMS

1. Let $y = \frac{1}{2}x^3 - x^2 + 1$. Find the values of y corresponding to $x = 2$ and $x = 2.5$. If x is first 2 and then 2.5, what are the values of Δx and Δy? Plot a small arc of the graph of the equation on a large scale and mark the segments corresponding to Δx and Δy.

2. In Problem 1 find the slope of the secant through the points whose abscissas are 2 and 2.5. Is this greater or less than the slope of the tangent at the point whose abscissa is 2? Is it a good approximation? How can a better one be obtained?

3. The equation of a curve is $y = 5 + 4x - x^2$. Find the slope of the secant through the points whose abscissas are 4 and 4.2. Plot a small arc of the curve on a large scale, draw the secant, and mark the segments corresponding to Δx and Δy.

4. If $xy = -6$, $x = 3$, and $\Delta x = 0.2$, calculate the values of $y + \Delta y$ and Δy. Illustrate by a graph.

5. Show that the distance formula of § 7 may be written in the form

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

6. Find the slope of the tangent to the given curve at the given point for each of the following equations. Find also the slope of the secant line for $\Delta x = 0.5$. In each case draw the curve, the tangent, and the secant line.

a. $y = x^2$, at $(-2, 4)$. *c.* $y = 2x - x^2$, at $(1, 1)$.

b. $y = x^2 - 3x$, at $(3, 0)$. *d.* $y = \frac{1}{x}$, at $(1, 1)$.

7. Find the exact slope of each of the following curves at the point indicated. In each case check the result by drawing a figure, showing the curve and the tangent *with the proper slope*.

a. $y = 2x^2 - x$, $x = 1$.
b. $y = x^2 - 7x + 3$, $x = 4$.
c. $y = x^3$, $x = 0$.
d. $y = \dfrac{x^2}{4}$, $x = 4$.
e. $x^2y = 24$, $x = 4$.
f. $y = x^2 - x - 2$, $x = 2$.
g. $xy = 12$, $x = 4$.
h. $y = 4 - x^2$, $x = 1$.

8. Find the inclination of the tangent to the curve at the given point in each case in Problem 7.

9. Find the slope of each of the following curves at *any* point $P(x, y)$. From this expression find the slope at the given points. In each case draw the curve and the tangents.

a. $y = x^2 - 3x$; $x = 0, 3$.
b. $y = x^3$; $x = -1, 0, +1$.
c. $y = 2x - x^2$; $x = 0, 1, 2$.
d. $y = x^2 + 5x + 4$; $x = -4, -1$.
e. $xy = -12$; $x = 2, 3, 4, 6$.
f. $y = \frac{1}{4}x^2 + 3x$; $x = 0, -6, -12$.
g. $y = 4x^2$; $x = -1, 0, +1$.
h. $y = 4 + 3x - x^2$; $x = 0, \frac{3}{2}, 3$.
i. $y = -\dfrac{6}{x}$; $x = -3, -2, -1$.
j. $y = x^3 - x^2 + 4$; $x = 0, \frac{2}{3}$.
k. $y = \dfrac{4}{x+2}$; $x = 0, 2, 4$.
l. $y = x^2$; $x = -2, -1, 0, 1, 2$.
m. $y = 8 - 2x^2$; $x = -2, -1, 0, 1, 2$.
n. $y = x^2 - 9$; $x = -3, -1, 0, 1, 3$.
o. $y = x^2 + 4x$; $x = -3, -1, 0, 1, 3$.
p. $y = x^3 - 3x$; $x = -2, -1, 0, 1, 2$.
q. $y = \dfrac{1}{x}$; $x = -2, -1, -\dfrac{1}{2}, \dfrac{1}{2}, 1, 2$.
r. $y = \dfrac{1}{x^2}$; $x = -2, -1, -\dfrac{1}{2}, \dfrac{1}{2}, 1, 2$.

39. Limits. In the previous section we obtained the slope of a tangent by regarding it as the limiting position of a secant. The idea of a variable approaching a constant as a limit occurs in elementary geometry, where the area of a circle is defined as the limit approached by the area of a regular circumscribed (or inscribed) polygon as the number of sides is increased indefinitely. Another example is found in the sum of the geometrical progression

$$S(n)=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\frac{1}{2^n},$$

where the sum S is a function of n, the number of terms taken. As the number of terms is increased indefinitely the sum approaches 2 as a limit.

In order to make definite the idea of a limiting value of a variable, we give the following definition:

The variable v is said to approach the constant l as a limit when the numerical value of the difference $v-l$ becomes and remains less than any fixed positive number, however small.

In applications of the preceding definition two variables are usually involved: (1) an independent variable which may be made arbitrarily to approach some constant as a limit, and (2) a dependent variable whose behavior is to be examined. If we call the independent variable z and the dependent variable v, which is a function of z, the question is to determine what constant value, if any, will be approached by v when z approaches some particular value a. If v approaches l when z approaches a, this fact is expressed by the notation

$$\lim_{z\to a} v = l,$$

read "The limit of v, as z approaches a, is l." For example, if $v=\cos z$, $\lim\limits_{z\to 0} v = \lim\limits_{z\to 0}\cos z = 1$.

40. Theorems on Limits. In finding the limits of functions, it is convenient to use the following theorems, which are here stated without proofs. In each theorem the variables mentioned are supposed to be functions of a single independent variable.

Theorem I. *The limit of an algebraic sum of any number of variables is equal to the same algebraic sum of their respective limits.*

Thus, if u, v, and w are functions of z, and if $u \longrightarrow A$, $v \longrightarrow B$, and $w \longrightarrow C$ as $z \longrightarrow a$, then

$$\begin{aligned}\lim (u + v - w) &= \lim u + \lim v - \lim w \\ &= A + B - C.\end{aligned}$$

It should also be noted that the theorem applies if one or more of the variables are replaced by constants. Thus, if $u = k$ and v is a function of z for which $\lim\limits_{z \to a} v = B$, then $\lim\limits_{z \to a} (k + v) = k + B$.

Theorem II. *The limit of the product of any number of variables is equal to the product of their respective limits.*

Thus, with the notation above,

$$\lim uvw = (\lim u)(\lim v)(\lim w) = ABC.$$

Also, if k denotes a constant,

$$\lim kv = k(\lim v) = kB.$$

From this theorem we may deduce also the following corollary: *The limit of the n^{th} power of a variable is the n^{th} power of the limit.*

Theorem III. *The limit of the quotient of two variables is equal to the quotient of their respective limits, provided the limit of the denominator is not zero.*

Thus, with the notation used above, $\lim\limits_{z \to a} \dfrac{u}{v} = \dfrac{A}{B}$, if $B \neq 0$. If $B = 0$ and $A \neq 0$, the quotient has no limit, but becomes

infinite as $z \rightarrow a$. If both A and B are zero, the limit is indeterminate, and can be evaluated only by some suitable transformation of the quotient (see example 2 below).

41. Evaluation of Limits. In applying the above theorems to the evaluation of the limit of a function, we meet with problems of varying degrees of difficulty. Let us consider some examples.

1. If $v = z^2$, the determination of the limit approached by v when z approaches any constant value a presents no difficulty, and we may write at once

$$\lim_{z \to a} v = a^2. \qquad \text{By Theorem II, § 40}$$

2. If $v = \dfrac{z^2 - 4}{z - 2}$, the determination of the limit approached by v when z approaches 2 cannot be made by direct application of Theorem III, because we should get $\frac{0}{0}$, a meaningless expression. The limit is best determined by transforming the expression as follows:

$$v = \frac{z^2 - 4}{z - 2} = \frac{(z + 2)(z - 2)}{z - 2} = z + 2.$$

It is now obvious that $\lim_{z \to 2} v = 4$. By Theorem I

If there is difficulty in understanding how the limit of a fraction can be definite, although both numerator and denominator approach zero, consider the adjoining figure. In this figure PR is tangent to the curve at P, and AC is perpendicular to PS at A. As A approaches P, B moves along PR, and C moves along the curve. Evidently both PA and AB approach zero, but their quotient, $\dfrac{AB}{PA}$, is a constant, being equal to $\tan RPS$.

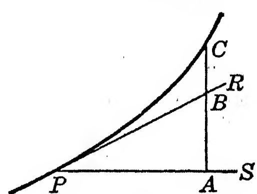


FIG. 50

Hence $$\lim \frac{AB}{PA} = \tan RPS.$$

3. If $v = \dfrac{z^2+4}{z-2}$, the determination of the limit approached by v when z approaches 2 presents another type of difficulty. Suppose that z starts with the value 3 and is made to decrease toward the value 2. Corresponding values of v are shown in the table. As z approaches nearer and nearer to the value 2, v becomes larger and larger and approaches no fixed number as a limit. Although in this case v approaches no limit, it is convenient to use the limit notation to express the fact that v increases indefinitely as z approaches 2 as a limit. This is done by writing

z	v
3.	13.
2.5	20.5
2.1	84.1
2.01	804.01
2.001	8004.001

$$\lim_{z \to 2} v = \infty.$$

This expression is often read "v becomes infinite when z approaches 2 as a limit." It should be clearly understood that v does not approach a limit and that the symbol ∞, called *infinity*, does not represent a number.

It is in this same sense that the expression $\dfrac{a}{0} = \infty$ is to be understood. It is impossible to divide a number by zero. But it is possible to divide a constant by successive values of a variable which approaches zero as a limit, in which case (if $a \neq 0$) we write

$$\lim_{z \to 0} \frac{a}{z} = \infty,$$

for which $\dfrac{a}{0} = \infty$ is merely an abbreviation frequently used.

The common expression "any number divided by zero is infinity" is used to mean "the quotient of any number

(different from zero) divided by a variable z becomes infinite when z approaches zero as a limit."

A similar meaning is to be attached to the phrase "$\tan 90° = \infty$"; namely that $\tan z$ increases indefinitely as z approaches $90°$ as a limit.

4. Examples in which the dependent variable approaches a limit when the independent variable approaches no limit, but increases indefinitely, are furnished by the definition of the area of a circle and by the sum

$$S(n) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$$

already cited. In the latter case, to express the fact that $S(n)$ approaches 2 as a limit when n increases indefinitely, the notation used is

$$\lim_{n \to \infty} S(n) = 2.$$

42. The Derivative. A study of the process of finding the slope of a curve at any point, as developed in § 38, will show that it may be regarded as a purely algebraic process, without reference to the geometrical problem which it is used to solve. As we shall see later, the limit of $\frac{\Delta y}{\Delta x}$ has many other uses, and for this reason it has been given a special name. We have, therefore, the following definition:

If one variable is a function of a second and if the increment of the first is divided by that of the second, the limit of the quotient, as the second increment approaches zero, is called the derivative of the first variable with respect to the second.

Thus, if $y = f(x)$, the limit of $\frac{\Delta y}{\Delta x}$, as Δx approaches 0, is the derivative of y with respect to x.

Notation. If $y = f(x)$, the derivative of y with respect to x is denoted by various symbols, the most common of which are

$$\frac{dy}{dx} \text{ or } dy/dx, \text{ read "}dy\text{ over }dx\text{";}$$

$$y', \text{ read "}y\text{ prime";}$$

$$f'(x), \text{ read "}f\text{ prime of }x\text{";}$$

$$\frac{d}{dx}f(x), \text{ read "}d\ dx\text{ of }f(x)\text{."}$$

Similarly, if $s = g(t)$, the derivative of s with respect to t is denoted by $\frac{ds}{dt}$, s', $g'(t)$, or $\frac{d}{dt}g(t)$.

Using this notation, we may write the above definition briefly as

$$y' = f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

Or, since $\Delta y = f(x + \Delta x) - f(x)$,

$$y' = f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

The symbol $\frac{dy}{dx}$ should not be regarded as a fraction; it is to be used as a single symbol, just as Δx is a single symbol.

43. Calculation of Derivatives. The process of finding the derivative is called *differentiation.* To differentiate one variable with respect to another we merely carry out the steps suggested by the definition.

I. *Let* $y = f(x)$. *Give* x *an increment* Δx *and, by substituting* $x + \Delta x$ *in the equation defining* y, *calculate* $y + \Delta y$, *the new value of* y.

II. *Subtract* y *from* $y + \Delta y$, *thus obtaining* Δy *in terms of* x *and* Δx.

III. *Divide* Δy *by* Δx, *thus obtaining the value of* $\dfrac{\Delta y}{\Delta x}$.

IV. *Find the limit of this expression as* $\Delta x \rightarrow 0$, *this being the value of* $\dfrac{dy}{dx}$.

Note that this is precisely the method used in § 38 to find the slope of a curve at any point. Furthermore, it should be clearly understood that during the process of differentiation x and y are regarded as having fixed values, while Δx and Δy vary and approach 0 as a limit. (Compare Example 1, § 38.)

EXAMPLE 1. Find $\dfrac{dy}{dx}$, if $y = x^3 - 2x + 4$.

Solution. Substituting $x + \Delta x$ for x, we have

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^3 - 2(x + \Delta x) + 4 \\ &= x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 2x - 2\Delta x + 4. \end{aligned}$$

Subtracting y from $y + \Delta y$,

$$\Delta y = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 2\Delta x.$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2 - 2.$$

Let $\Delta x \rightarrow 0$. The second and third terms of $\dfrac{\Delta y}{\Delta x}$ will evidently approach 0.

$$\frac{dy}{dx} = 3x^2 - 2.$$

EXAMPLE 2. If $f(t) = \dfrac{2}{t}$, find $f'(t)$.

Solution.
$$\begin{aligned} f(t) + \Delta f(t) &= f(t + \Delta t) \\ &= \frac{2}{t + \Delta t}. \end{aligned}$$

$$\Delta f(t) = \frac{2}{t + \Delta t} - \frac{2}{t}$$

$$= \frac{-2\,\Delta t}{t(t + \Delta t)}.$$

$$\frac{\Delta f(t)}{\Delta t} = \frac{-2}{t(t + \Delta t)}.$$

Let Δt approach 0, and we have

$$\frac{d}{dt} f(t), \text{ or } f'(t), = -\frac{2}{t^2}.$$

PROBLEMS

1. Find $\frac{dy}{dx}$ for each of the following:

a. $y = 4\,x^2.$

b. $y = 2 - x^2.$

c. $y = x^3 - x.$

d. $y = \frac{8}{x}.$

e. $y = 2\,x^3 - 3\,x^2.$

f. $y = \frac{4}{x - 2}.$

g. $y = x + \frac{1}{x}.$

h. $y = \frac{1}{x^2}.$

2. If $u = F(z)$, write four symbols for the derivative of u with respect to z.

3. Show that $\frac{d}{dx} x^4 = 4\,x^3.$

4. Find $\frac{ds}{dt}$ if $s = \frac{at}{t - a}.$

5. Find $\frac{ds}{dt}$ if $s = \frac{c}{t}.$

6. Find s' if $s = at^3 - bt^2.$

7. Find $f'(x)$ if $f(x) = ax^2 + bx + c.$

8. Find $F'(z)$ if $F(z) = \frac{z - 1}{z + 1}.$

9. Find $g'(z)$ if $g(z) = az - 2\,z^2.$

10. Find y' if $xy = 12$.

11. Find $\phi'(t)$ if $\phi(t) = \dfrac{1+t^2}{t}$.

12. Find $\dfrac{dy}{d\theta}$ if $y = \dfrac{\theta}{\theta + 1}$.

13. Find $\dfrac{dy}{dz}$ if $y = \dfrac{a}{z^2 + a^2}$.

44. The Derivative as a Slope. Using the notion of derivative, we can recast the method given in § 38 in the following theorem:

Theorem. *The slope of a curve at any point is the derivative of y with respect to x.*

Proof. Let $P(x, y)$ be any point on the curve and let Q be a neighboring point on the curve. The coördinates of Q are $x + \Delta x$, $y + \Delta y$. Hence the slope of the secant PQ is $\dfrac{\Delta y}{\Delta x}$.

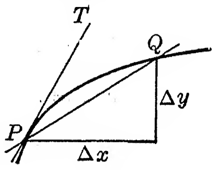


FIG. 51

By the definition of a tangent, the tangent line PT is the limiting position of PQ as $Q \rightarrow P$ or as $\Delta x \rightarrow 0$.

Therefore the slope of the tangent $= \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$.
But, by definition of derivative,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

Hence the slope of the tangent is $\dfrac{dy}{dx}$.

EXAMPLE. Find the slope of the curve whose equation is $x^2 + 2y = 8$ at the point where $x = 2$; also its inclination at that point.

Solution. Solving the equation for y in terms of x,

$$y = 4 - \frac{x^2}{2}.$$

Differentiating with respect to x,

$$y + \Delta y = 4 - \frac{x^2 + 2\,x\Delta x + (\Delta x)^2}{2},$$

$$\Delta y = \frac{-2\,x\Delta x - (\Delta x)^2}{2},$$

$$\frac{\Delta y}{\Delta x} = \frac{-2\,x - \Delta x}{2},$$

$$\frac{dy}{dx} = -\frac{2\,x}{2} = -x.$$

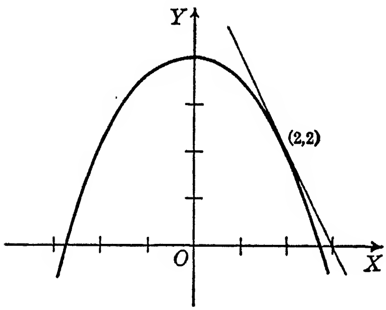


Fig. 52

Thus $-x$ is the slope at *any* point.

At the given point, $x = 2$; hence $m = -2$.

Since $m = \tan\alpha$, the inclination is the angle whose tangent is -2, or 116.57° (116° 34′). A carefully drawn figure verifies the result.

45. Differential Calculus. The student should not be misled by the theorem of § 44 into thinking that *derivative* is merely another name for *slope*. The use of the derivative in finding the slope of a curve is only one of many applications, of which others will be studied in later chapters. In fact, the study of the properties of derivatives and the solution of problems with their aid form a large part of that branch of mathematics known as *differential calculus.*

PROBLEMS

1. Find the slopes of the following curves at the points indicated, and in each case draw the curve and the tangent (or tangents) with the proper slope:

a. $y = x^3$; $x = 1, -1$.
b. $y = x^2 - 5x + 4$; $y = 0$.
c. $y = x^3 - 4x$; $x = 1$.
d. $xy = 12$; $x = -3$.
e. $y = 3 + 2x - x^2$; $x = 0$.
f. $y = (x-2)^3$; $x = 3$.
g. $y = \dfrac{8}{x+4}$; $x = 0, \pm 1$.

2. By setting $\dfrac{dy}{dx} = 0$, find the points at which the tangents to the following curves are parallel to the x-axis, and illustrate by a figure.

a. $4y = 16 + 12x - x^2$.
b. $6y = x^3 - 12x$.
c. $6y = 2x^3 - 3x^2 - 12x + 6$.
d. $xy - x^2 = 4$.

3. Write the equations of the tangents at the points given in Problem 1.

4. In Problem 1, *c*, find the slope of the secant through the points whose abscissas are 1 and 1.05, and compare your result with the one obtained by differentiating.

5. For the following equations tabulate the values of x, y, and m (the slope) corresponding to the values of x given; draw the tangents at the points indicated and *then* draw the curve.

a. $y = x^2 - 5x + 4$; $x = 0, 1, 2, 2.5, 3, 4, 5$.
b. $y = 3x + x^2$; $x = -4, -3, -2, -1.5, -1, 0, 1$.
c. $xy = -6$; $x = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 6$.
d. $8y = x^3$; $x = 0, \pm 1, \pm 2, \pm 3$.

6. Differentiate $y = \sqrt{x}$.

HINT. Rationalize the numerator of $\dfrac{\Delta y}{\Delta x}$.

CHAPTER IV

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS

46. The general method used in the previous chapter for finding a derivative is applicable in theory to any function; but in practice the calculation of a derivative by this means involves a large amount of labor, unless the function is of the simplest type. Fortunately it is possible to obtain formulas which may be used for differentiating all the functions ordinarily met in elementary mathematics.

In this chapter we shall derive formulas for the differentiation of algebraic functions; that is, functions involving sums, products, and powers of the independent variable x. Formulas for the differentiation of trigonometric, logarithmic, and exponential functions will be found in Chapters XIV and XV.

47. Derivative of a Power of x. Application of the general rule shows that

$$\frac{d}{dx}x^2 = 2x, \quad \frac{d}{dx}x^3 = 3x^2, \quad \frac{d}{dx}x^4 = 4x^3, \text{ etc.}$$

Now let $y = x^n$, where n is any positive integer. Applying the general rule of § 43, we have

I. $y + \Delta y = (x + \Delta x)^n$.

The second member may be expanded by the binomial theorem, giving

$$y + \Delta y = x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{1 \cdot 2}x^{n-2}(\Delta x)^2 + \cdots + (\Delta x)^n.$$

II. $\Delta y = nx^{n-1}\Delta x + \dfrac{n(n-1)}{1 \cdot 2} x^{n-2}(\Delta x)^2 + \cdots + (\Delta x)^n.$

III. $\dfrac{\Delta y}{\Delta x} = nx^{n-1} + \dfrac{n(n-1)}{1 \cdot 2} x^{n-2}(\Delta x) + \cdots + (\Delta x)^{n-1}.$

When Δx approaches zero as a limit, the limit of the first term on the right is nx^{n-1}, and the limit of each of the other terms is zero. Hence, by Theorem I, § 40,

IV. $\dfrac{dy}{dx} = nx^{n-1}.$

Hence

(I) $$\frac{d}{dx} x^n = nx^{n-1}.$$

Corollary. If $y = x$, $\dfrac{dy}{dx} = 1$. For in this case $n = 1$, and we have

$$\frac{d}{dx} x = 1\,x^0 \qquad \text{by Formula I}$$
$$= 1. \qquad \text{Since } x^0 = 1, \text{ by algebra}$$

This result we write as a separate formula

(I*a*) $$\frac{d}{dx} x = 1.$$

NOTE. This result could have been foreseen by considering the graph of $y = x$, which is a straight line of slope 1.

Formula I has been proved only for the case when n is a positive integer; but it is valid also when n has any constant value, positive or negative. For example, as may be verified by the general rule,

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx} x^{\frac{1}{2}} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

and $$\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{d}{dx} x^{-2} = -2x^{-3} = -\frac{2}{x^3}.$$

The proof of the formula when n is not a positive integer will be found in Chapter XIV.

48. Derivative of a Constant times a Power. The derivative of kx^n, where k is a constant, is k times the derivative of x^n. That this is true may be seen by reference to the proof in § 47. If we there replace $y = x^n$ by $y = kx^n$, we find that the effect is to multiply the right-hand side of each of the following equations by k. Hence step IV will become

$$\frac{dy}{dx} = knx^{n-1}, \quad \text{or} \quad \frac{d}{dx}kx^n = knx^{n-1}.$$

For example, $\frac{d}{dx}(6\,x^4) = 6 \cdot 4\,x^3 = 24\,x^3;$

$$\frac{d}{dx}(-4\,x^{-3}) = (-4)(-3)\,x^{-4} = 12\,x^{-4};$$

$$\frac{d}{dx}(-5\,x) = -5.$$

49. Derivative of a Constant. Suppose that $y = c$, where c is a constant. Applying the general rule, we get

I. $$y + \Delta y = c.$$

II. $$\Delta y = 0.$$

III. $$\frac{\Delta y}{\Delta x} = 0.$$

IV. $$\frac{dy}{dx} = 0.$$

Hence

(II) $$\frac{dc}{dx} = 0.$$

In words, *the derivative of a constant with respect to any variable is zero.*

This result could have been foretold by referring to the geometric interpretation of the derivative. The graph of $y = c$ is a straight line parallel to the x-axis, and the slope of this line is zero.

50. Derivative of a Sum of Functions. Let $y = u + v$, where u and v are functions of x. Applying the general rule, we get

I. $$y + \Delta y = u + \Delta u + v + \Delta v.$$

II. $$\Delta y = \Delta u + \Delta v.$$

III. $$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}.$$

When Δx approaches zero as a limit, the limit of the first term on the right is $\frac{du}{dx}$, and the limit of the second term is $\frac{dv}{dx}$. Hence, by Theorem I, § 40,

IV. $$\begin{aligned}\frac{dy}{dx} &= \lim \frac{\Delta y}{\Delta x}\\ &= \lim \frac{\Delta u}{\Delta x} + \lim \frac{\Delta v}{\Delta x}\\ &= \frac{du}{dx} + \frac{dv}{dx}.\end{aligned}$$

Therefore

(III) $$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

It is apparent that this proof can be extended to the case where y is an algebraic sum of any number of variables, and the formula may be stated in words as follows:

The derivative of an algebraic sum of any number of functions is equal to the same algebraic sum of their respective derivatives.

51. Summary and Examples. The methods and formulas of §§ 47–50 enable us to differentiate at sight any polynomial. We can also differentiate products of polynomials and fractions whose denominators consist of a single term, by first reducing them to the form of a polynomial.

We may therefore easily find the slope of any curve which has an equation that can be reduced to the following form: y equals a polynomial in x.

EXAMPLE 1. If $y = 5x^4 - 3x^2 + 6 + 2x^{-1}$, find $\frac{dy}{dx}$.

Solution. $\frac{dy}{dx} = \frac{d}{dx}(5x^4) - \frac{d}{dx}(3x^2) + \frac{d}{dx}(6) + \frac{d}{dx}(2x^{-1})$

$= 20x^3 - 6x - 2x^{-2}$. By Formula III

The derivatives of the first, second, and fourth terms were found by means of Formula I and § 48. The derivative of 6 is zero, by Formula II.

EXAMPLE 2. If $y = (2x + 3)(x^2 - 2)$, find $\frac{dy}{dx}$.

Solution. Multiplying the two polynomials, we have

$$y = 2x^3 + 3x^2 - 4x - 6.$$

Differentiating, as in Example 1, we find that

$$\frac{dy}{dx} = 6x^2 + 6x - 4.$$

In a later section we shall obtain a method of differentiating a product without multiplying the factors.

EXAMPLE 3. Find the slope of the tangents to the curve whose equation is $y = x + \frac{2}{x}$ at the points (2, 3) and (− 2, − 3). Draw a figure, showing the curve and the tangents.

Solution. Since the slope of the tangent to a curve at any point is $\frac{dy}{dx}$, we must differentiate $y = x + \frac{2}{x}$. This may be written in the form

$$y = x + 2x^{-1}.$$

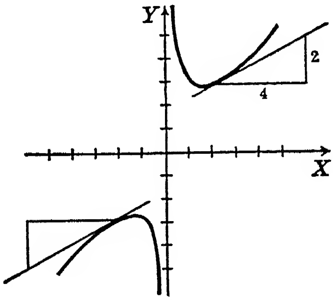


FIG. 53

Differentiating, we have
$$\frac{dy}{dx} = 1 - 2x^{-2}$$
$$= 1 - \frac{2}{x^2}.$$

Substituting $x = 2$ in the derivative, we find that the slope at $(2, 3)$ is $1 - \frac{2}{4} = \frac{1}{2}$. Similarly, the slope at $(-2, -3)$ is also $\frac{1}{2}$.

In drawing the tangent the method of § 12 should be used. Since the slope is $\frac{1}{2}$, this means that the tangent rises at the rate of $\frac{1}{2}$ vertical unit per horizontal unit, or 2 vertical units for each 4 horizontal units. Hence a second point on the tangent at $(2, 3)$ is $(6, 5)$.

PROBLEMS

1. Write the following functions in a column, and opposite each one write its derivative with respect to the independent variable: $5x^4$, $2x^{-3}$, $4\sqrt{x}$, $6x^{-\frac{1}{3}}$, $-7t$, πr^2, x^{-n}, $\frac{4}{3}\pi r^3$, $3z^{\frac{7}{3}}$.

2. Find $\frac{dy}{dx}$ for each of the following functions:

a. $y = 4x^5 - 2x^3 - 12.$

b. $y = 5x^7 - 3x^2 + 6x^{-1}.$

c. $y = 3x^2 - 5x.$

d. $y = 6 - \frac{2}{3}x^3.$

e. $y = 3x^{11} - 7x^9 + 11.$

f. $y = ax^2 + bx + c.$

3. Find

a. $\frac{d}{dx}(3x^{-2} - 2x^{-5}).$

b. $\frac{dz}{dx}$, if $z = x^3 - \frac{1}{x}.$

c. $f'(t)$, if $f(t) = 8t^5 + \frac{7}{t^3} - \frac{3}{t^4}.$

d. y', if $y = 1 + \frac{3}{t} - \frac{6}{t^2}.$

e. $\phi'(x)$, if $\phi(x) = -4x^{-5} + 6\sqrt{x}.$

4. Differentiate each of the following functions (in each case first reduce to the form of a polynomial by multiplication or division):

a. $y = \frac{x^3 - 1}{x^2}.$

b. $y = \frac{1 + x}{\sqrt{x}}.$

c. $f(t) = (t^2 - 1)^2.$

d. $\phi(t) = (t - 1)^3.$

e. $y = x(a + x)(a - x).$

f. $z = (u + 2)(2u^2 - 1).$

g. $z = \frac{6u^2 - 4u^3}{3u^2}.$

5 Differentiate the following functions, and in each case compute the value of the derivative for the given value of x:

a. $y=\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^3}$, $x=\frac{1}{2}$.

b. $y=\sqrt{x}+\frac{1}{\sqrt{x}}$, $x=4$.

c. $y=x^{\frac{3}{2}}+x^{-\frac{3}{2}}$, $x=9$.

d. $y=x^2-2+\frac{1}{x^2}$, $x=3$.

6. Find $\frac{dy}{dx}$ from each of the following equations.

HINT. First solve for y.

a. $x^2+2xy=6$.

b. $xy=18$.

c. $xy-2x=5$.

d. $\sqrt{x}+\sqrt{y}=1$.

e. $x^2y=-10$.

f. $xy^2=4$.

7. Find the slope of each of the following curves at the point indicated and draw an illustrative figure:

a. $y=\frac{x^2}{4}-3x$; $x=4$.

b. $3y=25x-x^2$; $x=10$.

c. $y=x^3-x^2-2x$; $x=2$.

d. $x^2+2xy=12$; $x=2$.

e. $x^2y=16$; $x=3$.

f. $8y=x^4-4x^2$; $x=3$.

g. $xy=12$; $x=2, 3, 4, 6$.

h. $y^2=8x$; $x=8$.

8. Differentiate $y=x^{-4}=\frac{1}{x^4}$ by the Δ-process. By the same method show that the derivative of x^{-n} is $\frac{-nx^{n-1}}{x^{2n}}=-nx^{-n-1}$.

9. Show by the Δ-process that if $y=x$, $\frac{dy}{dx}=1$.

10. Find a point on the graph of $y^2=8x$ where the inclination of the tangent is 45°.

11. Sketch the graph of $y=x^3$. Then find the slope of this curve at the origin and correct your graph accordingly.

12. Show by differentiation that the slope of the line whose equation is $y=mx+b$ is m.

52. Derivative of the Product of Two Functions. Let $y = uv$, where u and v are functions of x. When we give x an increment Δx, u and v will have increments Δu and Δv, since u and v are functions of x; hence y will have an increment, since it is the product of u and v. Therefore, when we apply the general rule, we have

I. $$y + \Delta y = (u + \Delta u)(v + \Delta v)$$
$$= uv + u\Delta v + v\Delta u + \Delta u\Delta v.$$

II. $$\Delta y = u\Delta v + v\Delta u + \Delta u\Delta v.$$

III. $$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}.$$

During the process of differentiating, the independent variable is supposed to have a definite fixed value x_0. Consequently the functions u, v, and y have corresponding definite fixed values. The quantities which vary during this process are the increments Δx, Δu, Δv, and Δy.

Consider separately the three terms of the second member. The first term consists of a product whose first factor does not vary when Δx approaches zero as a limit. The second factor approaches $\frac{dv}{dx}$ as a limit. Hence, by Theorem II, § 40, the first term approaches $u\frac{dv}{dx}$ as a limit.

Similarly, the second term approaches $v\frac{du}{dx}$ as a limit.

The first factor of the third term approaches zero as a limit, while the second factor approaches $\frac{dv}{dx}$. Hence, by Theorem II, § 40, the limit of the third term is zero. The third term might also have been written $\frac{\Delta u}{\Delta x}\Delta v$, from which the same result follows.

Finally, by Theorem I, § 40, we have

IV. $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Hence

(IV) $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

In words, *the derivative of the product of two functions is equal to the first function times the derivative of the second plus the second function times the derivative of the first.*

Example. Differentiate $y = (2x + 3)(x^2 - 2)$.

Solution. Here $u = 2x + 3$ and $v = x^2 - 2$. Hence $\frac{du}{dx} = 2$ and $\frac{dv}{dx} = 2x$. Applying Formula IV, we have

$$\begin{aligned}\frac{dy}{dx} &= (2x + 3)(2x) + (x^2 - 2)(2)\\ &= 4x^2 + 6x + 2x^2 - 4\\ &= 6x^2 + 6x - 4.\end{aligned}$$

Compare this result with that obtained in Example 2, § 51.

53. Derivative of the Product of a Constant and a Function. An important special case of Formula IV occurs when one factor is a constant. If $u = c$, $\frac{du}{dx} = 0$, by Formula II, and Formula IV becomes

(IV *a*) $$\frac{d}{dx}(cv) = c\frac{dv}{dx}.$$

In words this may be stated thus: *The derivative of a constant times a function is the constant times the derivative of the function.*

Note that the work of § 48 is a special case of Formula IV *a*.

54. Derivative of the Product of Several Functions. If y is the product of more than two functions, the derivative may be obtained by repeated application of Formula IV. Thus, if $y = uvw$, we may write

$$y = u(vw).$$

Hence
$$\frac{dy}{dx} = u\frac{d}{dx}(vw) + vw\frac{du}{dx}$$

$$= u\left[v\frac{dw}{dx} + w\frac{dv}{dx}\right] + vw\frac{du}{dx}.$$

Hence

(IV b)
$$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + vw\frac{du}{dx} + wu\frac{dv}{dx}.$$

A convenient form is obtained by dividing both sides of this formula by uvw. The result is

(IVc)
$$\frac{\frac{d}{dx}(uvw)}{uvw} = \frac{\frac{du}{dx}}{u} + \frac{\frac{dv}{dx}}{v} + \frac{\frac{dw}{dx}}{w}.$$

These results can be extended to the product of any number of functions. In actual differentiating, however, neither of the Formulas IV b and IV c is often used. It is usually better to reduce the given product to the form of the product of two functions only. For example, if the function is $y = x(a + x)(a - 3x)$, we transform this into $y = (ax + x^2)(a - 3x)$ before differentiating.

55. Derivative of the Quotient of Two Functions. Let $y = \frac{u}{v}$, where u and v are functions of x. Applying the general rule, we have

I.
$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}.$$

II. $$\Delta y = \frac{u+\Delta u}{v+\Delta v} - \frac{u}{v}$$

$$= \frac{uv + v\Delta u - uv - u\Delta v}{v^2 + v\Delta v}$$

$$= \frac{v\Delta u - u\Delta v}{v^2 + v\Delta v}.$$

III. $$\frac{\Delta y}{\Delta x} = \frac{v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}}{v^2 + v\Delta v}.$$

By Theorems I and II, § 40, the limit of the numerator as Δx approaches zero is

$$v\frac{du}{dx} - u\frac{dv}{dx},$$

and the limit of the denominator is v^2.

Hence, by Theorem III, § 40,

IV. $$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Hence

(V) $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

In words, *the derivative of the quotient of two functions is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

Special Case: the Denominator a Constant. If $v = k$, a constant, Formula V becomes

$$\frac{d}{dx}\left(\frac{u}{k}\right) = \frac{1}{k}\left(\frac{du}{dx}\right),$$

which is the same as Formula IV a, where c is replaced by $\frac{1}{k}$. Hence, to differentiate $\frac{u}{k}$, where k is a constant, write $\frac{u}{k}$ in the form $\frac{1}{k}(u)$ and use Formula IV a.

EXAMPLE 1. Differentiate $y = \dfrac{x^2+1}{x^3-3x}$.

Solution.

$$\frac{dy}{dx} = \frac{(x^3-3x)\dfrac{d}{dx}(x^2+1)-(x^2+1)\dfrac{d}{dx}(x^3-3x)}{(x^3-3x)^2}, \quad \text{by V}$$

$$= \frac{(x^3-3x)(2x)-(x^2+1)(3x^2-3)}{(x^3-3x)^2}$$

$$= \frac{-x^4-6x^2+3}{(x^3-3x)^2}.$$

EXAMPLE 2. Differentiate $y = \dfrac{4x^3-7x^2}{5}$.

Solution. This may be written as

$$y = \tfrac{1}{5}(4x^3-7x^2).$$

Hence, by Formula IV a, we have

$$\frac{dy}{dx} = \frac{1}{5}\frac{d}{dx}(4x^3-7x^2)$$

$$= \tfrac{1}{5}(12x^2-14x)$$

$$= \frac{12x^2-14x}{5}.$$

EXAMPLE 3. Differentiate $y = \dfrac{5}{4x^3-7x^2}$.

Solution. $$\frac{dy}{dx} = \frac{(4x^3-7x^2)(0)-5(12x^2-14x)}{(4x^3-7x^2)^2}, \quad \text{by V}$$

$$= \frac{-10(6x-7)}{x^3(4x-7)^2}.$$

PROBLEMS

1. Differentiate each of the following functions by means of Formula IV, and check your answers by obtaining the derivative according to the method of Example 2, § 51:

a. $y=(2x-1)(3x+2)$.

b. $y=(x^3-4)(2x^4-3)$.

c. $s=(t^2-3t)(t^3+4t^2)$.

d. $z=(u^3-2u)\left(1+\dfrac{2}{u}\right)$.

e. $f(x)=x(1-x)(1+x)$.

f. $y=(x^2-x+1)(x^2+x+1)$.

2. Differentiate each of the following functions:

a. $y=\left(x^6-\dfrac{1}{x}\right)\left(\dfrac{3}{x^2}+2x\right)$.

b. $y=\dfrac{x}{3-5x}$.

c. $y=\dfrac{x^2-1}{x^2-2x}$.

d. $y=\left(3x^{\frac{1}{3}}+1\right)\left(2x^{\frac{1}{2}}+1\right)$.

e. $f(x)=(x^2+2x)\left(\dfrac{1}{x^2}+\dfrac{2}{x}\right)$.

f. $s=\dfrac{1-t+t^2}{1+t+t^2}$.

g. $u=(t-1)(t+1)(t+2)$.

h. $u=\dfrac{z^3}{3-z^3}$.

i. $y=\dfrac{(1-x)(1+2x)}{1-3x}$.

j. $g(x)=\dfrac{x-\sqrt{x}}{x+\sqrt{x}}$.

k. $y=\dfrac{2-x^{\frac{1}{3}}}{2+x^{\frac{1}{3}}}$.

l. $s=\dfrac{a^2-t^2}{a^2+t^2}$.

m. $u=\dfrac{z^2}{z^2+a^2}$.

n. $x+y-2xy=0$.

o. $f(x)=\dfrac{7-\sqrt{x}}{7+\sqrt{x}}$.

p. $y=\dfrac{8a^3}{x^2+4a^2}$.

q. $y=\dfrac{-8}{1-x^4}$.

r. $y=\dfrac{x}{x^2+a^2}$.

3. Differentiate each of the following functions, and compute the value of the derivative for the indicated value of the independent variable:

a. $y=(x+4)\left(3-\dfrac{1}{\sqrt{x}}\right)$, $x=4$.

b. $s=\dfrac{t-4}{t+2}$, $t=2$.

c. $u=\dfrac{z}{\sqrt{z}+2}$, $z=9$.

d. $y=\dfrac{5}{5-x^2}$, $x=3$.

e. $xy=y+2$, $x=0$.

f. $y=\dfrac{\sqrt{x}+1}{\sqrt{x}-1}$, $x=7$.

which is the same as Formula IV a, where c is replaced by $\frac{1}{k}$. Hence, to differentiate $\frac{u}{k}$, where k is a constant, write $\frac{u}{k}$ in the form $\frac{1}{k}(u)$ and use Formula IV a.

EXAMPLE 1. Differentiate $y = \dfrac{x^2+1}{x^3-3x}$.

Solution.

$$\frac{dy}{dx} = \frac{(x^3-3x)\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x^3-3x)}{(x^3-3x)^2}, \quad \text{by V}$$

$$= \frac{(x^3-3x)(2x)-(x^2+1)(3x^2-3)}{(x^3-3x)^2}$$

$$= \frac{-x^4-6x^2+3}{(x^3-3x)^2}.$$

EXAMPLE 2. Differentiate $y = \dfrac{4x^3-7x^2}{5}$.

Solution. This may be written as

$$y = \tfrac{1}{5}(4x^3-7x^2).$$

Hence, by Formula IV a, we have

$$\frac{dy}{dx} = \frac{1}{5}\frac{d}{dx}(4x^3-7x^2)$$

$$= \tfrac{1}{5}(12x^2-14x)$$

$$= \frac{12x^2-14x}{5}.$$

EXAMPLE 3. Differentiate $y = \dfrac{5}{4x^3-7x^2}$.

Solution. $$\frac{dy}{dx} = \frac{(4x^3-7x^2)(0)-5(12x^2-14x)}{(4x^3-7x^2)^2}, \quad \text{by V}$$

$$= \frac{-10(6x-7)}{x^3(4x-7)^2}.$$

y is said to be a *function of a function.* We now proceed to derive a formula for obtaining $\frac{dy}{dx}$ in cases like this.

Let $y = f(u)$ and $u = g(x)$. In general, if x takes on an increment Δx, u will have an increment Δu, since u is a function of x. The increment Δu will cause y to have an increment Δy, since y is a function of u.

By elementary algebra, $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}$.

If Δx approaches zero, Δu also approaches zero; hence we may apply Theorem II, § 40, and the definition of a derivative, getting

$$\textbf{(VI)} \qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

EXAMPLE. Let $y = \sqrt[3]{4 + x - 3x^2}$. We may write this as $y = u^{\frac{1}{3}}$, where $u = 4 + x - 3x^2$. Hence Formula VI applies, and we have

$$\frac{dy}{du} = \frac{1}{3}u^{-\frac{2}{3}} = \frac{1}{3u^{\frac{2}{3}}}$$

and

$$\frac{du}{dx} = 1 - 6x.$$

Hence, by Formula VI, $\frac{dy}{dx} = \frac{1 - 6x}{3u^{\frac{2}{3}}}$.

A result in terms of x only may be obtained by substituting for u its value in terms of x.

57. Derivative of a Power of a Function. An important special case of Formula VI occurs when $y = u^n$, where n is a constant. Then, by Formulas I and VI,

$$\textbf{(VII)} \qquad \frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

This formula will be used constantly, and care must be taken to avoid the common error of forgetting the factor $\frac{du}{dx}$.

EXAMPLE 1. Differentiate $y = \sqrt{8x}$.

Solution. If this is written as $y = (8x)^{\frac{1}{2}}$, we see that y is of the form u^n, where

$$u = 8x, \quad n = \frac{1}{2}, \quad \text{and} \quad \frac{du}{dx} = 8.$$

Hence, by Formula VII,

$$\frac{dy}{dx} = \frac{1}{2}(8x)^{-\frac{1}{2}}(8)$$

$$= \frac{4}{\sqrt{8x}} = \frac{4}{2\sqrt{2x}} = \frac{\sqrt{2}}{\sqrt{x}}.$$

If we write the given expression as $y = 8^{\frac{1}{2}}x^{\frac{1}{2}}$, we can differentiate by means of Formulas I and IV a, and we obtain

$$\frac{dy}{dx} = (\sqrt{8})\frac{1}{2}x^{-\frac{1}{2}} = \frac{\sqrt{8}}{2\sqrt{x}} = \frac{2\sqrt{2}}{2\sqrt{x}} = \frac{\sqrt{2}}{\sqrt{x}},$$

as above. Note that neither method gives as a result $\frac{1}{2}(8x)^{-\frac{1}{2}}$, the result of the error alluded to above.

EXAMPLE 2. Differentiate $y = \sqrt{a^2 - x^2}$.

Solution. Here $u = a^2 - x^2$, $n = \frac{1}{2}$, and $\frac{du}{dx} = -2x$. Hence, by Formula VII,

$$\frac{dy}{dx} = \frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}(-2x)$$

$$= \frac{-x}{\sqrt{a^2 - x^2}}.$$

EXAMPLE 3. Differentiate $y = \dfrac{7}{(2x^2 - 3)^2}$.

Solution. This can be differentiated by the rule for a fraction; but when the denominator is a power and the numerator is a constant, it is better to use the power formula, as follows:

$$y = 7(2x^2 - 3)^{-2}.$$

$$\frac{dy}{dx} = -14(2x^2 - 3)^{-3}(4x)$$

$$= \frac{-56x}{(2x^2 - 3)^3}.$$

58. The Derivative as an Aid in Curve-Sketching. The fact that the slope of a curve at any point is given by the value of the derivative at that point is of considerable assistance in quickly obtaining a fair sketch of a curve. Usually, if the points where the tangents are parallel to the x-axis are found, a very good idea of the general shape of the curve can be obtained by finding two or three more points. This is because the points where the tangents are horizontal are usually (not always) the highest or lowest points of the curve in that neighborhood. The slope at points of intersection with the x-axis is sometimes a help.

EXAMPLE 1. Find the points where the tangent to the curve whose equation is $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 2$ is parallel to the x-axis, and sketch the curve.

Solution. Since the tangent is to be parallel to the x-axis, we must find the points where the slope is zero. Differentiating, we have

$$\frac{dy}{dx} = x^2 - x - 2.$$

Setting this equal to 0, we obtain

$$x^2 - x - 2 = 0;$$

whence $\quad (x-2)(x+1) = 0,$

or $\quad x = -1 \text{ and } x = 2.$

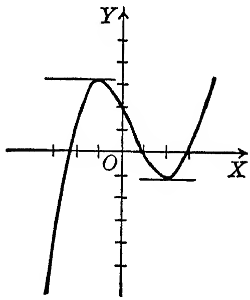


FIG. 54

Substituting these values in the given equation, $y = \frac{19}{6}$ and $y = -\frac{4}{3}$. Hence the required points are $(-1, \frac{19}{6})$ and $(2, -\frac{4}{3})$. Three other points easily obtained are $(0, 2)$, $(3, \frac{1}{2})$, and $(-3, -\frac{11}{2})$. Since there are but two points where the tangent is horizontal, the curve must rise continuously at the right of the point $(3, \frac{1}{2})$ and fall at the left of the point $(-3, -\frac{11}{2})$.

EXAMPLE 2. Find the points where the tangents to the curve whose equation is $y = x\sqrt{2 - x^2}$ are parallel to the x-axis, and sketch the curve.

Solution. Writing this in the form $y = x(2 - x^2)^{\frac{1}{2}}$ and differentiating, we have

$$\frac{dy}{dx} = \frac{x}{2}(2 - x^2)^{-\frac{1}{2}}(-2x) + (2 - x^2)^{\frac{1}{2}}(1)$$
$$= \frac{2(1 - x^2)}{(2 - x^2)^{\frac{1}{2}}}.$$

Setting the derivative equal to 0, we obtain $x = -1$ and $+1$. The corresponding values of y are -1 and $+1$.

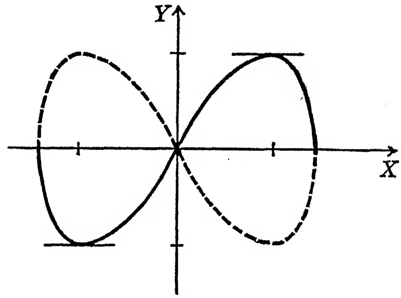


FIG. 55

From the given equation we find that the x-intercepts are 0 and $\pm\sqrt{2}$. For the latter values, $\dfrac{dy}{dx} = \dfrac{-2}{0}$; hence at these points the slope is infinite and the tangents are perpendicular to the x-axis.

In the figure the dotted line shows the graph of

$$y = -x\sqrt{2 - x^2}.$$

PROBLEMS

1. Find $\dfrac{dy}{dx}$ for each of the following functions:

a. $y = u^3,\ u = 2 + \sqrt{x}.$

b. $y = \sqrt{u} - u,\ u = x^2 + 2x.$

c. $y = u^4 - u^2,\ u = 2x^3 - x + 1.$

d. $y = \dfrac{u}{u - 2},\ u = \dfrac{2x - 3}{2x + 3}.$

2. Differentiate each of the following functions:

a. $y = \sqrt{4 + 3x}.$

b. $y = (5 - 2x)^3.$

c. $y = \left(x + \dfrac{1}{x}\right)^5.$

d. $y = \sqrt{7 - 2x + x^2}.$

e. $y = \dfrac{1}{\sqrt{a^2 - x^2}}.$

f. $y = (2 - 3x)^{\frac{3}{2}}.$

g. $y = \sqrt{1 + x^3}.$

h. $y = \dfrac{1}{\sqrt{x^2 + 4x + 10}}$.

i. $y = \sqrt{3x} - \dfrac{1}{\sqrt{3x}}$.

j. $f(t) = \sqrt{at^2 - t^3}$.

k. $z = (4 - u^2)^3$.

l. $g(z) = \left(\dfrac{z}{1-z}\right)^2$.

3. Differentiate each of the following functions in two ways and show that the answers thus obtained are identical:

a. $y = (3 + 4x)^3$.

b. $y = \sqrt{2t} + (2t)^{\frac{3}{2}}$.

4. Find $\dfrac{dy}{dx}$ in each of the following equations:

a. $x^2 + y^2 = 9$.

b. $b^2x^2 + a^2y^2 = a^2b^2$.

c. $y^2 = 2px$.

d. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

5. Find the points where the following curves have horizontal tangents, and sketch the curve in each case:

a. $y = 3x - x^3$.

b. $y = x^2(x-2)^2$.

c. $y = \dfrac{x^4}{4} - 2x^2 - 1$.

d. $y = (x-3)^3$.

e. $y^2 = x^3$.

f. $y = (x-1)(x+1)(x-2)$.

6. Show that the graph of any equation which has the form $y = a + bx + cx^2$ has one point where the tangent is parallel to the x-axis.

7. Find the inclination of the tangent to the circle whose equation is $x^2 + y^2 = 20$ at the point (2, 4).

8. Find the slope of the curve whose equation is $x^2 - 4y^2 = 20$ at the point (6, 2) and write the equation of the tangent.

59. Inverse Functions. From the definition of a function it follows that if $y = f(x)$, then x is a function of y, and this may be written $x = g(y)$. In simple cases the second relation may be obtained by solving the equation connecting the variables for x in terms of y. For example, if

$$x^2 - y - 1 = 0,$$

then

$$y = f(x) = x^2 - 1,$$

while

$$x = g(y) = \pm\sqrt{y+1}.$$

Such functional relations are called *inverse*.

When it is easier to get x as a function of y than to get y as a function of x, $\frac{dy}{dx}$ may be found by using the formula

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}.$$

To prove this relation observe that, by algebra,

$$\frac{\Delta y}{\Delta x}=\frac{1}{\frac{\Delta x}{\Delta y}}.$$

As Δx approaches zero, Δy in general approaches zero also. Hence, if we pass to the limit, we obtain

(VIII)
$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}.$$

Also

(VIII *a*)
$$\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}.$$

That is, *the derivative of a function is the reciprocal of the derivative of the inverse function.*

This may be verified, for example, in the case of the functions mentioned above. If

$$y=x^2-1,$$

then
$$\frac{dy}{dx}=2x;$$

and if
$$x=\sqrt{y+1},$$

then
$$\frac{dx}{dy}=\frac{1}{2\sqrt{y+1}}.$$

It is obvious that $2x$ and $\frac{1}{2\sqrt{y+1}}$ are reciprocals.

60. Formulas for the Differentiation of Algebraic Functions. With the formulas derived in the previous sections any algebraic function may be differentiated. The most important ones are collected here for convenience.

$$\text{(I)} \qquad \frac{d}{dx}x^n = nx^{n-1}.$$

$$\text{(I}a\text{)} \qquad \frac{d}{dx}x = 1.$$

$$\text{(II)} \qquad \frac{d}{dx}c = 0.$$

$$\text{(III)} \qquad \frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$$

$$\text{(IV)} \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

$$\text{(IV }a\text{)} \qquad \frac{d}{dx}(cv) = c\frac{dv}{dx}.$$

$$\text{(V)} \qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

$$\text{(VI)} \qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

$$\text{(VII)} \qquad \frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

$$\text{(VIII)} \qquad \frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}.$$

The following examples illustrate the differentiation of several types of functions relatively more complicated than those given before.

EXAMPLE 1. Find the derivative of $y = \sqrt{3 + 4\sqrt{x}}$.

Solution. This may be written in the form

$$y = (3 + 4\sqrt{x})^{\frac{1}{2}}.$$

$$\frac{dy}{dx} = \frac{1}{2}(3 + 4\sqrt{x})^{-\frac{1}{2}}\frac{d}{dx}(4\sqrt{x}) \qquad \text{by Formula VII}$$

$$= \frac{1}{2}(3 + 4\sqrt{x})^{-\frac{1}{2}}(2x^{-\frac{1}{2}}) \qquad \text{by Formulas I and IV}a$$

$$= \frac{1}{(3 + 4\sqrt{x})^{\frac{1}{2}}}\frac{1}{x^{\frac{1}{2}}} = \frac{1}{\sqrt{3x + 4x^{\frac{3}{2}}}}.$$

EXAMPLE 2. Differentiate $y = x\sqrt{2 - 3x}$.

Solution. $$y = x(2 - 3x)^{\frac{1}{2}}.$$

$$\frac{dy}{dx} = x\frac{d}{dx}(2 - 3x)^{\frac{1}{2}} + (2 - 3x)^{\frac{1}{2}}\frac{d}{dx}x \qquad \text{by Formula IV}$$

$$= x\frac{1}{2}(2 - 3x)^{-\frac{1}{2}}(-3) + (2 - 3x)^{\frac{1}{2}}$$

by Formulas VII and IVa

$$= \frac{-3x}{2\sqrt{2 - 3x}} + \sqrt{2 - 3x}$$

$$= \frac{4 - 9x}{2\sqrt{2 - 3x}}. \qquad \text{By simplification}$$

EXAMPLE 3. Differentiate $y = \dfrac{x^2}{\sqrt{x^2 + a^2}}$.

Solution. $$\frac{dy}{dx} = \frac{\sqrt{x^2 + a^2}\dfrac{d}{dx}x^2 - x^2\dfrac{d}{dx}\sqrt{x^2 + a^2}}{x^2 + a^2} \qquad \text{by Formula V}$$

$$= \frac{\sqrt{x^2 + a^2} \cdot 2x - x^2\dfrac{1}{2}(x^2 + a^2)^{-\frac{1}{2}}\dfrac{d}{dx}(x^2 + a^2)}{x^2 + a^2}$$

by Formulas I and VII

$$= \frac{2x\sqrt{x^2 + a^2} - \dfrac{x^2 \cdot 2x}{2\sqrt{x^2 + a^2}}}{x^2 + a^2} \qquad \text{by Formula I}$$

$$= \frac{x^3 + 2a^2x}{(x^2 + a^2)^{\frac{3}{2}}}. \qquad \text{By simplification}$$

PROBLEMS

1. Differentiate the following functions:

a. $s = 4t^2 - 4\sqrt{t}$.

b. $x = \dfrac{a+t^2}{a-t^2}$.

c. $y = x^2\sqrt{3-x}$.

d. $u = t\sqrt{4+t^2}$.

e. $f(x) = x\sqrt{7-3x}$.

f. $s = \dfrac{t}{\sqrt{1+2t}}$.

g. $y = \sqrt{\dfrac{a+x}{a-x}}$.

h. $u = \dfrac{v}{\sqrt{1-v^2}}$.

i. $y = \dfrac{3+4x}{\sqrt{4-3x^2}}$.

j. $y = (3-x^3)^{\frac{1}{3}}(2+x^2)^{\frac{1}{2}}$.

2. In the following problems find the value of $\dfrac{dy}{dx}$ for the given value of x:

a. $y = x^2 + \dfrac{4}{x^2}$, $x = 2$.

b. $y = \dfrac{x}{3-x}$, $x = 2$.

c. $y = \sqrt{7x} + 7\sqrt{x} + \dfrac{1}{\sqrt{7x}} + \dfrac{7}{\sqrt{x}}$, $x = 7$.

d. $y = \dfrac{x^2}{a^2-x^2}$, $x = 0$.

e. $y = \sqrt{a^2-x^2}$, $x = a$.

3. In the following equations find $\dfrac{dy}{dx}$:

a. $x = \sqrt{y} + \sqrt[3]{y^2}$.

b. $x = \sqrt{4+y^2-y^4}$.

c. $x = 3\sqrt{4-y^2}$.

4. Draw the graphs in *a*, *b*, *d*, and *e* of Problem 2, showing the tangent at the point indicated. In each case check the graph by finding the slope at two other points.

5. Show that the slope at any point of the curve whose equation is $y^2 = 2px$ is $\dfrac{p}{y}$. Illustrate by a graph.

6. Show that the slope at any point of the curve whose equation is $x = ay^3$ is $\frac{1}{3\,ay^2}$. Illustrate by a graph.

7. Find the point at which the tangent to the curve whose equation is $x = a + by + cy^2$ is perpendicular to the x-axis. Take $a = b = c = 1$ and draw a figure.

61. Implicit Functions. Let a relation between x and y be expressed by the equation

$$x^2 + y^2 - a^2 = 0. \tag{1}$$

This equation may be solved for y, in which case $y = \sqrt{a^2 - x^2}$ is called an *explicit* function of x, or it may be solved for x, in which case $x = \sqrt{a^2 - y^2}$ is called an explicit function of y. If the equation is not solved but stands in the original form (1), it is said that y is an *implicit* function of x, or x is an implicit function of y.

A derivative may be calculated from either the implicit or explicit form of the function. Suppose that, in the example above, it is desired to find the derivative of y with respect to x.

If the explicit form is used, $y = \sqrt{a^2 - x^2}$,

whence
$$\frac{dy}{dx} = \frac{-x}{\sqrt{a^2 - x^2}}. \tag{2}$$

If the implicit form is used, it is necessary only to differentiate the terms of equation (1) as they stand *with respect to x*. The result is

$$2x + 2y\frac{dy}{dx} = 0;$$

whence, by solving,
$$\frac{dy}{dx} = -\frac{x}{y}. \tag{3}$$

The results in (2) and (3) are easily seen to be identical.

In the example above there is little choice between the explicit and implicit forms of the function. In some cases the implicit form is more convenient.

Consider, for example,

$$x^2 + 3\,xy + y^2 - 4\,x + 8\,y - 12 = 0.$$

It is possible to solve this equation and to get an explicit expression for y as a function of x. It is far more convenient, however, to calculate $\frac{dy}{dx}$ from the implicit form. Thus, differentiating the terms as they stand, we have

$$2\,x + (3\,x\frac{dy}{dx} + 3\,y) + 2\,y\frac{dy}{dx} - 4 + 8\frac{dy}{dx} = 0.$$

Solving this equation for the derivative gives

$$\frac{dy}{dx} = \frac{4 - 2\,x - 3\,y}{8 + 3\,x + 2\,y}.$$

In still other cases it is impossible to solve the equation for either y or x, and the implicit form must be used to calculate the derivative. For example,

$$x^6 + xy + y^6 - 3 = 0$$

cannot be solved algebraically. The derivative of y with respect to x can be calculated, however, by differentiating the terms as they stand.

Thus, $$6\,x^5 + \left(x\frac{dy}{dx} + y\right) + 6\,y^5\frac{dy}{dx} = 0;$$

whence, solving for the derivative,

$$\frac{dy}{dx} = -\frac{6\,x^5 + y}{x + 6\,y^5}.$$

Note that when the derivative is found by the implicit method, it is usually a function of y as well as of x.

EXAMPLE. Find the slope of the tangent to the ellipse whose equation is $x^2 + 2y^2 = 24$ at the points where $x = 4$.

Solution. Differentiating implicitly, we have

$$2x + 4y\frac{dy}{dx} = 0,$$

whence
$$\frac{dy}{dx} = -\frac{x}{2y}.$$

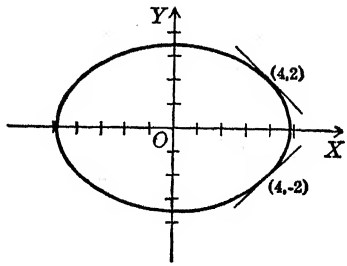


FIG. 56

Since the derivative involves both x and y, we must find the value of y from the given equation. When $x = 4$, $y = \pm 2$. Substituting in the expression for $\frac{dy}{dx}$, we find that the slope at (4, 2) is -1 and the slope at $(4, -2)$ is $+1$.

62. Successive Differentiation. If y is a function of x, (that is, if $y = f(x)$) the derivative is also a function of x.

$$\frac{dy}{dx} = y' = f'(x).$$

This new function of x can be differentiated with respect to x, giving $\frac{dy'}{dx}$, which is called the *second derivative* of y with respect to x. This process can be continued, and by successive differentiation we obtain the *third* derivative, *fourth* derivative, etc.

Notation. If $y = f(x)$, the following notations are used for the derivatives:

First derivative, $\quad \frac{dy}{dx} = y' = f'(x).$

Second derivative, $\quad \frac{d^2y}{dx^2} = y'' = f''(x).$

Third derivative, $\quad \frac{d^3y}{dx^3} = y''' = f'''(x).$ Etc.

EXAMPLE 1. Find the successive derivatives of

$$f(x) = x^3 - 2x^2 + 6.$$

Solution.

$$f'(x) = 3x^2 - 4x.$$

$$f''(x) = 6x - 4.$$

$$f'''(x) = 6.$$

$$f^{iv}(x) = 0.$$

Evidently all the derivatives from this point on are zero.

EXAMPLE 2. Find the second derivative of $y = \sqrt{a^2 - x^2}$.

Solution. The first derivative is

$$\frac{dy}{dx} = \frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{a^2 - x^2}}.$$

Differentiating again, we have

$$\frac{d^2y}{dx^2} = \frac{\sqrt{a^2 - x^2}(-1) + x\dfrac{-x}{\sqrt{a^2 - x^2}}}{a^2 - x^2} = \frac{-a^2}{(a^2 - x^2)^{\frac{3}{2}}}.$$

PROBLEMS

1. Find $\frac{dy}{dx}$ in the following equations:

a. $x^2 + xy - y^5 = 0.$

b. $x^2y - xy^2 + y^3 - 9 = 0.$

c. $y^3 + xy + x + y = 0.$

d. $x^2 + 3xy + y^2 = 31.$

e. $x^2 + \sqrt{xy} + y^2 = 1.$

f. $x + \sqrt{xy} + y = 6.$

g. $(2x)^{\frac{2}{3}} + (9y)^{\frac{2}{3}} = 13.$

h. $x^3 + y^3 - 3axy = 0.$

2. Show that the slope at any point of the ellipse whose equation is $b^2x^2 + a^2y^2 = a^2b^2$ is $-\frac{b^2x}{a^2y}$.

3. Show that the slope at any point of the hyperbola whose equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{b^2x}{a^2y}$.

4. Show that the slope at any point of the circle whose equation is $x^2 + y^2 = a^2$ is $-\dfrac{x}{y}$.

5. The locus of the equation $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is a parabola whose axis of symmetry is the line $x = y$. Show that the slope at any point is $-\sqrt{\dfrac{y}{x}}$. Plot the locus for $a = 9$, taking values of x which are perfect squares and using the double sign before $\sqrt{x}$. Check the graph by noting the slope at various points.

6. The locus of the equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is called a hypocycloid of four cusps. Show that its slope at any point is $-\sqrt[3]{\dfrac{y}{x}}$. Sketch the graph by finding the intercepts and the points where $x = \pm y$, and the slope at each of these points.

7. Find the intercepts of the curve whose equation is $y^{\frac{2}{3}} + x^2 = a^2$. By differentiating implicitly, show that the slope at each of the corresponding points is 0, and sketch the curve.

8. Find the slope of the tangent to the curve whose equation is $x^3 + y^3 - 3\,xy = 0$ at the point where $x = y$. Draw the tangent at this point and a small arc of the curve.

9. Find the second derivatives of the following functions:

a. $y = 4\,x^5 - 2\,x^3 - 12.$

b. $y = 3\,x^2 - 5\,x.$

c. $f(x) = 3\,x^{-2} - 2\,x^{-3}.$

d. $f(t) = 8\,t^5 + \dfrac{7}{t^3} - \dfrac{3}{t^4}.$

e. $\phi(x) = -4\,x^{-5} + 6\sqrt{x}.$

f. $y = \dfrac{x^3 - 1}{x^2}.$

g. $f(t) = (t^2 - 1)^2.$

h. $y = x(a + x)(a - x).$

i. $z = (u + 2)(2\,u^2 - 1).$

j. $y = \dfrac{1}{x} + \dfrac{1}{x^2} - \dfrac{1}{x^3}.$

k. $y = x^{\frac{3}{2}} + x^{-\frac{3}{2}}.$

l. $s = t^2 - 2 + \dfrac{1}{t^2}.$

63. Applications of the Derivative in Geometry. It has been shown that the slope or direction of a curve at any point can be found by getting the value of the derivative $\dfrac{dy}{dx}$

at that point. Other problems which are based upon this fundamental idea are the following:

1. To find the points on a curve where the tangent is horizontal; that is, parallel to the x-axis.

2. To find the points where a curve has a given direction, or where the tangent is parallel to a given line.

3. To find the angle of intersection between two curves. *By the angle of intersection between two curves is meant the angle between their tangents at a point of intersection.*

4. To find the equation of the tangent and the equation of the normal to a curve at a given point. *The normal at any point on a curve is the line perpendicular to the tangent at that point.*

The first of these problems was illustrated in § 58; the following examples illustrate the others:

EXAMPLE 1. Find the points on the circle whose equation is $x^2 + y^2 = 20$, where the tangents are parallel to the line whose equation is $2x + y = 0$.

Solution. Differentiating the equation of the circle, we get

$$2x + 2y\frac{dy}{dx} = 0;$$

whence $$\frac{dy}{dx} = -\frac{x}{y}.$$

Since the slope of the given line is -2, we set

$$-\frac{x}{y} = -2;$$

whence $$x = 2y.$$

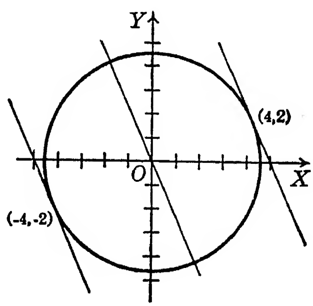


FIG. 57

Substituting $x = 2y$ in the equation of the circle, we find that $y = \pm 2$. The corresponding values of x are ± 4. The proper pairing of these values is determined from the value of the derivative to be $(4, 2)$ and $(-4, -2)$.

EXAMPLE 2. Find the angles of intersection between the curves whose equations are $y = x^2$ and $y = 2x$.

Solution. Solving the equations simultaneously, the points of intersection are found to be (0, 0) and (2, 4). The slope of the first curve at any point is given by $\frac{dy}{dx} = 2x$. The second curve is a straight line, and its slope at every point is 2.

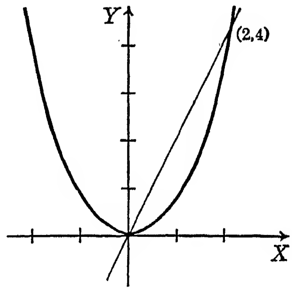


FIG. 58

Angle at (0, 0). The inclination of the straight line is 63.43° (63° 26′). The inclination of the curve at (0, 0) is 0. Hence, by subtraction, the angle of intersection at the origin is 63.43°.

Angle at (2, 4). For $x = 2$ the slope of the curve is 4 and its inclination is 75.96° (75° 58′). The inclination of the straight line is 63.43° (63° 26′). Hence, by subtraction, the angle of intersection at (2, 4) is 12.53° (12° 32′).

EXAMPLE 3. Find the equations of the tangent and of the normal to the ellipse whose equation is $x^2 + 4y^2 = 8$ at the point (2, 1).

Solution. We first make sure that the point (2, 1) lies on the curve. Differentiating the equation of the curve, we have

$$2x + 8y\frac{dy}{dx} = 0,$$

whence $$\frac{dy}{dx} = -\frac{x}{4y}.$$

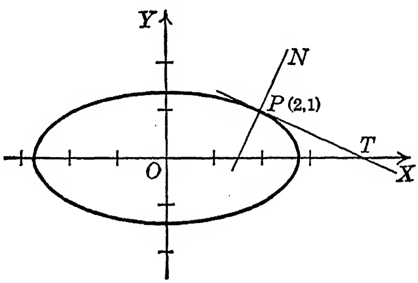


FIG. 59

The value of the derivative at (2, 1) is $-\frac{1}{2}$. Hence the slope of the tangent is $-\frac{1}{2}$ and the slope of the normal is 2.

Using the point-slope form, the equation of the tangent PT is

$$y - 1 = -\tfrac{1}{2}(x - 2), \quad \text{or} \quad x + 2y - 4 = 0,$$

and the equation of the normal PN is

$$y - 1 = 2(x - 2), \quad \text{or} \quad 2x - y - 3 = 0.$$

If the equation of a tangent with a given slope is required, we first determine the point or points of tangency, as in Example 1.

PROBLEMS

1. Find the points on the following curves where the tangent is horizontal. Draw the figure in each case.

a. $y = 3x + x^2$.
b. $y = 4 - x - x^2$.
c. $y = 3x - x^3$.
d. $y = 2x^2 - x^4$.
e. $y = x^3 + 1$.
f. $y = x^3 + x$.

2. Find the points on the curve $y^2 = x$ where the tangent is parallel to the line $x - 4y - 4 = 0$.

3. Find the points on the curve $4y = x^2$ where the inclination is 45°; 135°; 60°.

4. Find the points on the curve $y = 3x^2 + x^3$ where the tangent is perpendicular to the line $4x - 9y - 18 = 0$.

5. Find the equations of the tangent and of the normal to the following curves at the points indicated:

a. $y = x^2 - 2x$, at $(0, 0)$.
b. $y = x^2 - 4$, at $(1, -3)$.
c. $x^2 + y^2 = 10$, at $(-1, 3)$.
d. $x^2 + 4y^2 = 20$, at $(2, -2)$.
e. $x^2 - y^2 = 7$, at $(4, 3)$.
f. $x^2 - 2y^2 + 4 = 0$, at $(-2, 2)$.

6. Find the length of the tangent to the curve $y = x^2 - 6x$ at $(2, -8)$ from the point of tangency to the point where it cuts the x-axis.

7. In Problem 6 find the length of the normal between the curve and the y-axis.

8. In Problem 5, *d*, find the length of the segment of the tangent intercepted by the coördinate axes.

9. Find the angles of intersection of the following pairs of curves:

a. $y^2 = 2x$, $x - y - 4 = 0$.

b. $2y = x^2 - 4$, $8y = x^2 - 4$.

c. $x^2 + y^2 = 17$, $3x - 5y + 17 = 0$.

d. $x^2 + 9y^2 = 36$, $x + 3y = 6$.

e. $x^2 + y^2 = 36$, $x^2 = 5y$.

f. $x^2 = 4y$, $y = \dfrac{8}{4 + x^2}$.

g. $x^2 + y^2 = 25$, $xy = 12$.

10. Show that the curves whose equations are $xy = k$ and $x^2 - y^2 = k$ intersect at right angles.

11. Show that the curve whose equation is $y = Ax^3 + Bx^2 + Cx + D$ has not more than two points at which its tangent is horizontal. Show that the tangent is horizontal at no point if $B^2 < 3AC$.

12. Show that the tangents to the parabola $y^2 = 2px$ at the points where $x = \frac{p}{2}$ are perpendicular to each other.

CHAPTER V

FUNCTIONS — MAXIMUM AND MINIMUM VALUES

64. The work of the previous chapters has been mainly geometrical. We have learned to represent curves and straight lines by equations and to solve certain geometrical problems by means of these equations. In particular, the introduction of derivatives has made it possible to find the slope of a curve at any point, to find where the tangent to a curve has any given direction, and to find the angle between two curves at a point of intersection — problems not solvable by other means. We now turn to the study of certain problems which furnish a wider scope for the employment of the methods developed in Chapters I–IV, but which are not primarily geometrical.

65. Functions in General. Many of the problems of mathematics involve the study of quantities of varying magnitude. We shall confine ourselves for the present to the case where there are two variables of which one is a function of the other.

In Chapter III the term *function* was defined as follows: *If one of two variables has one or more definite values corresponding to each value given to the other variable, the first variable is a function of the second.* We there used the notation "$y=f(x)$" to represent an equation connecting x and y; and when x and y had concrete meanings, they were coördinates of a point tracing the graph of the equation $y=f(x)$.

It is clear, however, that the definition is capable of much wider applications. The variables may represent any

quantities, and we may know that one variable is a function of another (in the sense of the definition) without thinking of the equation connecting them, or, indeed, without even knowing any such equation. For example, the velocity of a falling body is a function of the distance through which it has fallen; the premium paid on a $1000 life-insurance policy is a function of the age of the insured; the normal rate of coal consumption of a locomotive is a function of the speed.

In examples like those just stated the functional law is sometimes given merely by a table of values of the two variables. For example, it is found by experiment that the temperature $\theta°$ of a vessel of cooling water t minutes after the beginning of the observation is as given in the adjoining table. Evidently θ is a function of t, and decreases when t increases. The limitations of this method of stating a functional relation are equally evident: for values of t not given in the table we can only approximate values of θ by interpolation if t is less than 20, and we cannot do even this if t is greater than 20.

t	θ
0	92.0
1	85.3
2	79.5
3	74.5
5	67.0
7	60.5
10	53.5
15	45.0
20	39.5

66. Derivation of Equations of Functions. For mathematical purposes it is desirable, when possible, to express the functional relation by an equation or formula connecting the variables. The advantages of the equation are (1) that for each value of one variable we can calculate the value of the other as accurately as we please; (2) that we can memorize the equation, if we have occasion to use the law frequently; (3) that from the equation, as we shall see presently, we can deduce many properties of the law of variation.

To find the equation connecting two variables, we must know the way in which they are related, and translate this relation into algebraic language. In many cases this can be done by means of the rules of algebra and geometry, as in the derivation of the mensuration formulas of geometry and trigonometry. The examples below illustrate methods available at this point; more difficult problems are reserved for more advanced processes of calculus.

EXAMPLE 1. The parcel-post regulations prescribe that the sum of the length and the girth of a package must not exceed 84 in. Rectangular boxes with square ends which come just within the rule are to be constructed. Express the volume of such a box as a function of the side of the square end.

Solution. Let x be the length of the side of the square, l the length of the box, and V the volume. By geometry,

$$V = lx^2.$$

But the regulations prescribe that

$$l + 4x = 84.$$

Hence we may substitute $84 - 4x$ for l. This gives

$$V = x^2(84 - 4x),$$

or

$$V = 84x^2 - 4x^3,$$

which is the desired solution.

EXAMPLE 2. Experiments in an artesian well showed that the temperature increased with the depth at the rate of $4° C$ per hundred meters. If the average temperature at the surface was $12° C$, express the temperature below the surface as a function of the depth.

Solution. Let T be the temperature in degrees and d the depth in meters. Since T increased at the rate of 4° per hundred meters, it increased at the rate of 0.04° per meter. Hence at a depth of d meters T had increased $0.04\,d$ degrees. But at the surface, $T = 12$. Thus we have the formula

$$T = 12 + 0.04\,d.$$

67. Variation. In scientific books we sometimes find laws connecting two variables stated like the following: "The extension (E) of a coiled spring *varies as* the weight (W) supported." The phrase "y varies as x" means simply that $y = kx$, where k is some constant. For example, the statement above means that $E = kW$, where k is a constant. In like manner the area of a circle varies as the square of the radius, for $A = \pi R^2$.*

One quantity is said to "vary inversely"† as another when the first is equal to a constant times the reciprocal of the second. Thus, if y varies inversely as x, $y = \frac{k}{x}$, or $xy = k$, where k is a constant.

In either case the value of the constant can be found from a knowledge of a pair of corresponding values of the variables. It obviously depends not only on the nature of the variables but also on the units of measurement employed.

Example. By Boyle's law the pressure of a gas at constant temperature varies inversely as the volume. It is found that a certain quantity of gas of volume 24 cu. in. exerts a pressure of 30 lb. per square inch. Find the equation connecting the pressure and the volume.

Solution. Let p be the pressure and v the volume. Since p varies inversely as v, the desired equation will be of the form $p = \frac{k}{v}$ or $pv = k$. To find k, substitute the given values of p and v. This gives

$$24 \cdot 30 = k, \quad \text{or} \quad k = 720.$$

Therefore the required equation is

$$pv = 720.$$

* The phrase "varies as" must be distinguished from the more general phrase "varies with." If y is any function of x, y varies *with* x, but it varies *as* x only if $y = kx$. Thus the area of a circle varies *with* the radius, but varies *as* the square of the radius.

† "Inversely" because as x increases, y decreases, and vice versa.

PROBLEMS

1. Express the area of a square as a function of its diagonal.

2. Express the radius of a sphere as a function of (*a*) its volume, (*b*) its surface.

3. A rectangle is inscribed in a circle of radius 10. Express the area of the rectangle as a function of (*a*) one side, (*b*) the angle between a side and a diagonal.

4. In a circle of radius 10 express the length of a chord as a function of its distance from the center. Tabulate the various lengths of the chord for integral values of its distance from the center.

5. Express the volume of a cylindrical parcel just satisfying the parcel-post regulation as a function of its radius. Calculate the values of the volumes corresponding to radii 6 in., 8 in., and 10 in.

6. The cost of setting the type for a pamphlet is $500, and the charge for paper and printing is 50¢ per copy. Express the cost per copy ($\$C$) as a function of the number (n) of copies printed.

7. A closed cylindrical can is to hold 58 cu. in. Express the amount of material (M) required to make it as a function of the radius (r). Calculate this amount for radii 1.5 in., 2 in., and 2.5 in.

8. A road rises with a slope m. If y is the increase in altitude per thousand feet measured along the road, express y as a function of m.

9. The boiling-point of water is 212° F. at sea level and decreases at the rate of 2° per thousand feet of increase in altitude. Express the boiling-point as a function of the height in feet.

10. The pressure of wind on a vertical plane surface varies as the square of its velocity. If the pressure per square foot is 1 lb. when the velocity is 15 mi. per hour, express the pressure as a function of the velocity.

11. In t seconds a body falls a distance of s feet, where $s = \frac{1}{2}gt^2$. Its velocity v after t seconds is gt. Express v as a function of s. (Here g is a constant, approximately equal to 32.)

12. A rectangle with sides parallel to the axes is inscribed in the curve whose equation is $4x^2 + 9y^2 = 36$. Express the area of this rectangle as a function of x.

13. A ball is thrown upward with an initial velocity of 100 ft. per second. This velocity decreases at the rate of 32 ft. per second per second. Express the velocity (v) as a function of the time the ball is in motion (t). When does the ball reach its highest point?

14. Show that if y varies as x, any two values of y are proportional to the corresponding values of x.

15. Above the surface of the earth the weight of a body varies inversely as the square of its distance from the center of the earth. If an object weighs 100 lb. at the surface of the earth, express its weight as a function of its distance from the center of the earth, and find its weight when it is 100 mi. above the surface of the earth. (Take the radius of the earth as 4000 mi.)

16. The time required for one swing of a pendulum varies as the square root of its length. If the time required for one swing of a pendulum whose length is 3.25 ft. is 1 sec., express the time as a function of the length. Also find the length of a pendulum which makes one swing in 0.5 sec.

68. Graphs of Functions. Some of the properties of a function can be studied by means of the graph of the equation giving the functional relation. This graph is plotted like the graphs of equations of loci.

It is customary to take the horizontal axis as the axis of the independent variable, and the vertical axis as that of the dependent variable. If the values of one variable are so much larger than those of the other that it is difficult

to draw a smooth curve through the plotted points, use different scales for the variables. In every case the scales used must be indicated on the axes.

It should be noted that when the functional relation is given merely by a table of values, we are saved the labor of calculating this table. But it is impossible to improve our graph by calculating intermediate values, as we can do when the equation is given.

EXAMPLE 1. It is found that when weights are suspended from a certain coiled spring, the length of the spring increases at the rate of 0.12 in. per pound; when no weight is suspended, its length is 4 in. Express the length (l inches) of the spring as a function of the weight (w pounds) suspended, and plot the graph of this function.

Solution. Since l increases at the rate of 0.12 in. per pound, the original length of 4 in. will be increased by $0.12\,w$ inches when w pounds are suspended. Hence the desired equation is

$$l = 4 + 0.12\,w.$$

Since this equation is of the first degree, its graph is a straight line, and two pairs of values are sufficient.

w	l
0	4
10	5.2

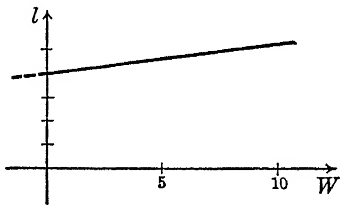


FIG. 60

An interpretation of the graph is interesting and important. First of all, we note that the part of the graph at the left of the l-axis is dotted, since negative values of w have no meaning. Second, if we compare the above equation with the

slope-intercept form of the straight-line equation, $y = mx + b$, we see that the coefficient of w, which is 0.12, is the slope of the line. But 0.12 is the rate at which the length of the spring increases with respect to the weight suspended. Hence the rate of change of l with respect to w is merely the slope of the graph.

EXAMPLE 2. Plot the graph of $V = 84x^2 - 4x^3$.

Solution. This is the function obtained in Example 1, § 66. The table of values should not extend beyond the points where x and V cease to have a meaning. In this case neither variable can be negative, and the last pair of values serves only to determine the point at which the curve crosses the x-axis. The large values of V make it necessary to use different scales for x and V, in order to get the curve on the paper.

x	V
0	0
2	304
4	1068
6	2160
8	3328
10	4400
12	5184
14	5448
16	5120
18	3888
20	1600
22	-- 1936

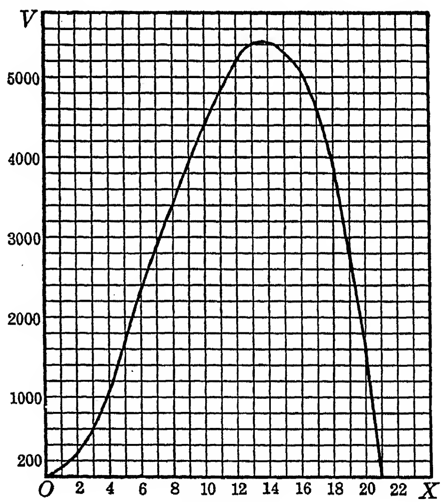


FIG. 61

Inspection of the graph reveals several things. First, V increases slowly at the beginning, and then rapidly until $x = 12$; then it decreases rapidly for $x > 16$. The graph appears to cross the x-axis at $x = 21$, a fact easily verified by substitution

in the equation. Second, the largest value of V appears to be that for $x = 14$, but the student must not make the mistake of thinking that the largest plotted value is the largest possible value. Perhaps V is still larger for some value of x near 14, like 14.01. Third, from the figure we may read off values of V corresponding to any values of x, or vice versa. For example, when $V = 3000$, $x = 7.5$ or 19, nearly.

69. The Graph as a Computing Diagram. One of the most elementary uses of the graph of a function is reading off from the graph approximate values of one variable corresponding to those of the other. Since these are usually between the calculated values by means of which the curve has been plotted, this process is frequently called *graphical interpolation.* For example, we may read off from the figure of Example 2 of the previous section the following approximate results: when $x = 11$, $V = 4800$; when $x = 13$, $V = 5400$; when $x = 11.5$, $V = 5000$; when $V = 1000$, $x = 3.8$ or 20.4; when $V = 2000$, $x = 5.7$ or 19.7; etc.

In order that a graph may be used effectively for this purpose, it should be fairly large,* and the scales on the horizontal and vertical axes must be properly chosen. The scale along either axis should be one of the following: 1 space = 1 unit, 2 units, or 5 units; 10 units, 20 units, or 50 units; 0.1 unit, 0.2 unit, or 0.5 unit; etc. The purpose in choosing one of these is to facilitate the estimation of an additional decimal place. Furthermore, the scales should be such that the curvature of the graph is plainly shown; that is, no part of the graph should be a straight line (unless, of course, the equation is linear). As remarked in the previous section, the scales used must be marked on the axes.

* For each problem on pages 137–140 a sheet of paper measuring about 6 in. by 8 in. should be used.

EXAMPLE. In the last figure the plotting-paper used measured 24 spaces by 29 spaces. Since the interval of variation of x was from 0 to 21, the scale used on the x-axis was 1 space $=1$ unit of x. The interval of variation of V was from 0 to 5500. This made the largest possible scale on the V-axis 1 space $= 200$ units, requiring 28 spaces. With this diagram only two-figure accuracy is possible; but with a somewhat larger figure and careful estimation three-figure accuracy is possible.

The graph is most useful as a computing diagram when it is necessary to approximate a large number of values of one of the variables, and it is used chiefly for this purpose. When only one value is desired it is easier to calculate it directly than to plot the graph.

70. Graphical Interpolation compared with Interpolation by Proportional Parts. Let us plot the graph of $y=x^3$, x ranging from 0 to 2. From the graph we see that the value of y for $x=1.6$ is very nearly 4. By computation we find that it is exactly 4.096.

If we try to find the value of $(1.6)^3$ by interpolating in the table of values by proportional parts, we obtain the very erroneous result 5.2. The process of interpolation by proportional parts is as follows. We have given the table

x	$y=x^3$
1	1
1.6	?
2	8

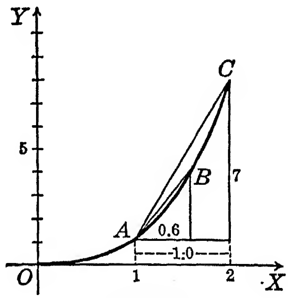


FIG. 62

In this we are to find the missing value of y. If we use the increment notation, we have the following equations:

When $\Delta x = 2 - 1 = 1$, $\Delta y = 8 - 1 = 7$.

When $\Delta x = 1.6 - 1 = 0.6$, find Δy.

We assume that corresponding values of Δx and Δy are proportional;

that is, $$\frac{\Delta y}{0.6} = \frac{7}{1}, \text{ whence } \Delta y = 0.6(7) = 4.2,$$

and $$y = 1 + 4.2 = 5.2.$$

Referring to the figure, we see that this requires that the slope of AB be the same as that of AC. This is true in general only if the graph is a straight line. Hence we conclude that

Interpolation by proportional parts is accurate only if the segment of the graph between the adjacent values of the variable is nearly a straight line.

PROBLEMS

1. In 1906 the intercollegiate track records were as follows:

d	100 yd.	220 yd.	440 yd.	880 yd.
t	9.8 sec.	21.2 sec.	48.8 sec.	1 min. 56 sec.

d	1 mi.	2 mi.
t	4 min. 17.8 sec.	9 min. 27.6 sec.

Exhibit this information graphically, plotting t as a function of d. Do the parts of the curve between the plotted points have any meaning? Explain.

2. A body is thrown vertically up with an initial velocity of 100 ft. per second. Its distance from the ground after t seconds is s feet, where $s = 100t - 16t^2$. Plot the graph of this function and estimate from the graph when the body reaches its greatest height.

3. A rectangle has an area of 320 sq. rd. Express the perimeter as a function of the length of one side and plot the graph. From the graph estimate the smallest value of the perimeter.

4. The radius of a circle is 6, and a tangent is drawn to it from a point x units distant from the center of the circle. Express the length of the tangent as a function of x and plot this function. From the graph estimate the values of x for which the length of the tangent is 12; 18.

5. The sag of a trolley wire is given by the formula

$$s = \sqrt{\frac{3\,d(l-d)}{8}},$$

where $s =$ the sag, $l =$ the length of the wire, and $d =$ the distance between poles. If $d = 80$, plot the graph of this function, letting l vary from 80 to 160.

6. In a class numbering 50 the grades were distributed as follows:

Grade	95	90	85	80	75	70	65	60	55	50
Number	1	2	6	9	5	10	6	4	4	3

Exhibit this information graphically.

7. Out of 100,000 children of age 10, mortality tables show that the following numbers live to the age specified:

Age (x)	10	20	30	40	50
Number (n)	100,000	92,637	85,441	78,106	69,804

Age (x)	60	70	80
Number (n)	57,917	38,569	14,474

Plot n as a function of x. Estimate from the graph how many live to be 52 years old; to what age a child 10 years old has an even chance of living.

8. The observed temperatures ($\theta°$) of a vessel of cooling water at times t minutes from the first observation were

t	0	1	2	3	5	7	10	15	20
θ	92.0	85.3	79.5	74.5	67	60.5	53.5	45	39.5

Plot θ as a function of t. Estimate from the graph at what time the temperature became 50°; what the temperature was 18 min. after the beginning of the observations.

9. A rectangle is inscribed in a circle of diameter 5. Express the area of the rectangle as a function of one side, plot the graph of this function, and determine from the graph what value of the side makes the area greatest.

10. The answer to Problem 5 on page 131 is $V = 84\pi r^2 - 2\pi^2 r^3 = 264 r^2 - 19.7 r^3$. Plot the graph of this function, taking $r = 0$, 2, 4, 6, 8, 10, 12, and 14. From the graph determine the maximum volume possible, and also the radii giving volumes of 3000, 4000, 5000, and 6000 cubic inches.

11. The monthly charge for gas in a certain city is $0.50 in addition to $1.20 per thousand cubic feet actually consumed. Express the cost per thousand cubic feet as a function of the number of thousand cubic feet consumed, and plot the graph of this function.

12. The relation between Fahrenheit temperatures (F) and centigrade temperatures (C) is linear. When $F = 212°$, $C = 100°$; and when $F = 32°$, $C = 0°$. Express F as a function of C and plot the graph. At what rate does F increase per degree centigrade? From the graph read off the values of C corresponding to $F = -20$, 0, 20, 40, 60, 80, and 100.

13. It is desired to construct an open-topped rectangular box, with a square base, which will hold just 6 cu. ft. Express the amount of material required as a function of the side of the base. Plot the graph of this function and estimate from the graph the dimensions which will require the least material.

14. The dimensions of a rectangular block are 2 in., 4 in., and 8 in. If each dimension is decreased by x inches, what is the volume remaining? Plot the graph of this function and determine from the graph what value of x will make the remaining volume just one half the given volume.

15. With a certain crane it is found that the force F (measured in pounds) which will just overcome a weight w is given by the formula $F = 4.21 + 0.043\,w$. Plot the graph of this function. What meanings can be attached to the constants 4.21 and 0.043?

16. A box is to be made from a piece of cardboard 10 in. by 15 in. by cutting a square of side x from each corner and turning up the sides. Express the volume in terms of x. Plot the graph of this function and from the graph find the values of x which correspond to $V = 60$, 70, 80, 90, 100, 110, 120, and 130 cu. in.

17. Express the area of a rectangle inscribed in a semicircle of radius 8 as a function of the base. Plot the graph of this function and find from the graph for what value of the base the area is greatest.

18. Express the perimeter of the rectangle of Problem 17 as a function of the base and determine from the graph of the function the value of the base which makes the perimeter a maximum.

19. The range (R) in thousands of yards of a projectile fired with a certain muzzle velocity at an angle of elevation (θ) expressed in mils* is given by the formula $\theta = 5\,R(3 + R)$. Plot θ as a function of R, letting R vary from 0 to 6 and using a large scale for R. From the graph tabulate the ranges corresponding to elevations 0, 50, 100, etc., up to 300 mils.

71. Increasing and Decreasing Functions. ***Definition.*** *A function $y = f(x)$ is said to be an* ***increasing*** *function if y increases (algebraically) when x increases. The function is* ***decreasing*** *if y decreases when x increases. Obviously a function may be increasing for some values of x and decreasing for others.*

This characteristic of a function appears in its graph. Let the figure be the graph of $y = f(x)$. The graph shows that the function decreases when x increases from $x = a$

* A mil is a unit of angular measurement used in artillery problems; 1600 mils = 90°.

to $x = b$, and from $x = c$ to $x = d$; and that the function increases when x increases from $x = b$ to $x = c$. The graph shows also that the slope of the tangent is positive when the function is increasing, and negative when the function is decreasing. Since the slope of the tangent is equal to the value of the derivative, we have the following criterion:

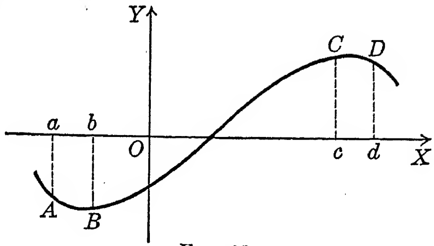


FIG. 63

A function is increasing when its derivative is positive, and decreasing when its derivative is negative.

EXAMPLE 1. For what values of the variable x is the function $V = 84x^2 - 4x^3$ increasing, and for what values is it decreasing?

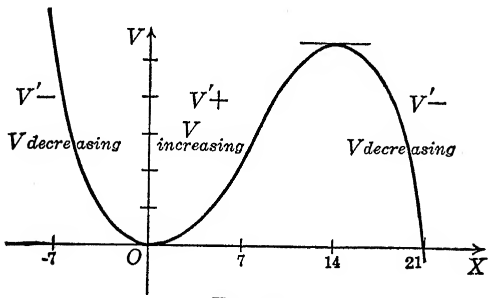


FIG. 64

Solution. This is the function considered in some detail in Example 2, § 68. Differentiating,

$$\frac{dV}{dx} = 168\,x - 12\,x^2 = 12\,x\,(14 - x).$$

The derivative is written in *factor form* so that we may find its sign by considering the sign of each factor. The derivative

is zero when $x = 0$ and when $x = 14$. We now test, in succession, values less than 0, between 0 and 14, and greater than 14.

When $x < 0$, $12x$ is $-$ and $14 - x$ is $+$. Hence $\frac{dV}{dx}$, which is the product of these factors, is $-$, and V is decreasing.

When $x > 0$ and at the same time < 14 (usually written $0 < x < 14$), $12x$ is $+$ and $14 - x$ is $+$. Hence $\frac{dV}{dx}$ is $+$, and V is increasing.

When $x > 14$, $12x$ is $+$ and $14 - x$ is $-$. Hence $\frac{dV}{dx}$ is $-$, and V is decreasing.

These results are shown on the graph.

EXAMPLE 2. For what values of x is the function $y = 3x - x^3$ increasing, and for what values is it decreasing?

Solution. The solution, which is similar to that in Example 1, may be condensed as follows:

Differentiating,
$$\begin{aligned} y' &= 3 - 3x^2 \\ &= 3(1 + x)(1 - x). \end{aligned}$$

Hence $y' = 0$ when $x = -1$ and when $x = +1$.

Testing the sign of y', we have the following results:

For $x < -1$, $\quad y' = 3(-)(+) = -$.

For $-1 < x < +1$, $\quad y' = 3(+)(+) = +$.

For $x > +1$, $\quad y' = 3(+)(-) = -$.

Hence the function is increasing for all values of x between -1 and $+1$; for values of x less than -1 or greater than $+1$ it is decreasing.

PROBLEMS

1. Is the given function increasing or decreasing for the given values of x?

a. $y = \dfrac{x}{1 + x^2}$, $x = 0, 2$.

b. $y = x\sqrt{x^2 + 1}$, $x = 0, 2$.

c. $y = x^2 + \dfrac{1}{x}$, $x = -1, +1$.

d. $y = \sqrt[3]{x^2 + 2}$, $x = -1, 0, +1$.

2. For what values of x are the following functions increasing? In each case illustrate your answer by a sketch of the graph.

a. $y = x^3 - 2.$ *c.* $y = x + \frac{4}{x}.$

b. $y = x - x^2.$ *d.* $y = 2x^3 - 3x^2 - 12x.$

3. Can a function be neither increasing nor decreasing for a single value of x? Give an illustration.

4. Is the function $y = 4x - x^2$ increasing or decreasing when $x < 2$? when $x > 2$? when $x = 2$? Explain the meaning of your last answer.

5. Is the function $y = x^5 + 1$ increasing or decreasing when $x = 0$? Explain your answer.

6. In each of the following equations determine the values of the independent variable for which the function is increasing and those for which it is decreasing:

a. $s = 100t - 16t^2.$ *d.* $A = x^2 + \frac{24}{x}.$

b. $s = \sqrt{30(l - 80)}.$ *e.* $F = 4.21 + 0.043w.$

c. $V = 264r^2 - 19.7r^3.$ *f.* $V = 4x^3 - 50x^2 + 150x.$*

EXTREME VALUES

72. A Problem. Suppose that it is required to construct a rectangular bin, with a square base, to have a capacity of 100 cu. ft. How many square feet of lumber will be required? Obviously there is no definite answer to this question, because a bin with a square base and capacity of 100 cu. ft. can be constructed in many different ways.

If the base is made 10 ft. square, the depth would have to be 1 ft. This would require 100 sq. ft. for the base and 10 sq. ft. for each side, making a total of 140 sq. ft.

* These functions are taken from Problems 2, 5, 10, 13, 15, and 16, respectively, on pages 137–140.

If the base is made 5 ft. square, the depth would have to be 4 ft. This would require 25 sq. ft. for the base and 20 sq. ft. for each side, making a total of 105 sq. ft.

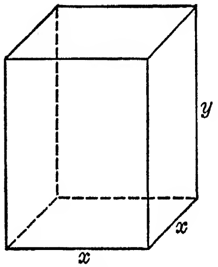


FIG. 65

If the base is made 2 ft. square, the depth would have to be 25 ft. This would require 4 sq. ft. for the base and 50 sq. ft. for each side, making a total of 204 sq. ft.

The amount of material required depends on the size of the base, which can be chosen arbitrarily. If A represents the amount of material required and x is the edge of the base, then A is a function of x. To find a mathematical expression for this function we proceed as follows:

If y is the depth of the bin, then

$$A = x^2 + 4xy.$$

This is the surface of any rectangular bin. But the volume of this one was to be 100 cu. ft.; hence

$$x^2y = 100.$$

The latter equation shows that $y = \frac{100}{x^2}$; and, by substituting this value in the former equation, we get

$$A = x^2 + \frac{400}{x}.$$

This functional relation enables us to calculate easily the amount of material required when the size of the base has been selected. Some results are shown in the table. Beginning at the top, the table shows that as the values of x increase, the values of A decrease until x reaches the value 6, after which the values of A increase.

Some of the bins require much less material than others. *What is the least amount from which the bin can be built?*

The table indicates that the least amount of material is $102\frac{2}{3}$ sq. ft. It is unlikely, however, that this result is exact. The values of x were selected at random, and there is no reason why they should all be integers. It is quite probable that A will be less than $102\frac{2}{3}$ when x has some value between 5 and 6 or between 6 and 7. But to attempt to find by experiment the exact value of x for which A will have its least value is a hopeless task. (Compare the discussion of Example 2, § 68.) In order to find the exact value of x for which A will have its least value, the methods of the calculus are employed, as explained in the following paragraphs.

Edge of base $= x$ feet	Material $=$ A square feet
1	401
2	204
3	$142\frac{1}{3}$
4	116
5	105
6	$102\frac{2}{3}$
7	$106\frac{1}{7}$
8	114
9	$125\frac{4}{9}$
10	140

If we draw the graph of the function $A = x^2 + \dfrac{400}{x}$, it becomes evident that at the point on the graph corresponding to the smallest value of A the tangent will be horizontal. Hence we have merely to find the value of x for which the slope of the curve is zero. The slope at *any* point, however, is the value of the derivative. Therefore *the value of x for which A is least is that for which the derivative is zero*, or

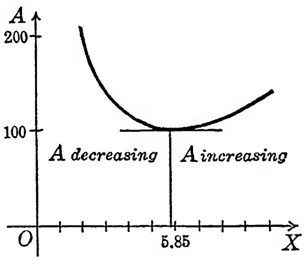


FIG. 66

$$\frac{dA}{dx} = 2x - \frac{400}{x^2} = 0.$$

Solving this equation for x, we obtain in succession

$$2x^3 - 400 = 0,$$

$$x^3 = 200,$$

and $$x = \sqrt[3]{200} = 5.85 \text{ (approximately)}.$$

Substituting this value in the expression for A, we obtain $A = 34.22 + 68.38 = 102.60$ sq. ft.

73. Maxima and Minima. Critical Values. ***Definitions.*** *A maximum value of a function is one that is greater than any value just preceding or following. A minimum value of a function is one that is less than any value just preceding or following.* Maximum and minimum values are called collectively *extreme values.*

More precisely, if $y = f(x)$, the value of y for $x = a$ is a maximum if it is larger than the values of y corresponding to values of x a little smaller and also a little larger than a. Similarly, the value of y for $x = a$ is a minimum if it is smaller than the values of y corresponding to values of x a little smaller and also a little larger than a.

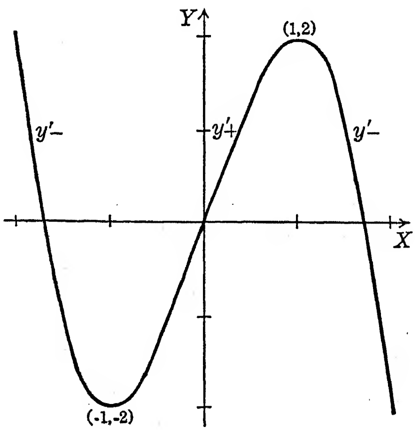


FIG. 67

EXAMPLE. Consider the function $y = 3x - x^3$. (See also Example 2, § 71.) This function has a maximum value, $y = 2$, for $x = 1$; and a minimum value, $y = -2$, for $x = -1$. These are shown in the adjoining graph, which indicates that the values of y for all values of x a little different from 1 are *less*

than 2, while the values of y for all values of x a little different from -1 are *greater* than -2. It should be noticed that the point corresponding to a maximum value is not necessarily the highest point on the graph of the function, but that it is higher than any point very near it on the graph. A similar statement holds for a minimum value.

The principle involved in finding maxima and minima may be explained in two ways, both of which must be understood.

1. The definition shows that when a function reaches a maximum value, it ceases to increase and begins to decrease. That is, if $y = f(x)$ has a maximum value for $x = a$,

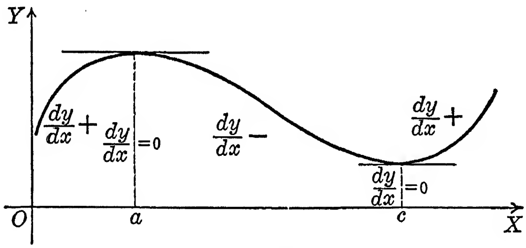


FIG. 68

y is increasing for $x < a$ and decreasing for $x > a$. But in § 71 we learned that if a function is increasing, its derivative is positive, and if it is decreasing, its derivative is negative. Hence $\frac{dy}{dx}$ is $+$ when $x < a$ and $-$ when $x > a$. Therefore, at $x = a$, $\frac{dy}{dx}$ changes from $+$ to $-$, and therefore $\frac{dy}{dx} = 0$ at $x = a$. The analagous explanation for the case when y has a minimum value for $x = a$ is left to the student.

2. Referring to the above figure, it is apparent that at either a maximum (as at $x = a$) or at a minimum (as at

$x = c$) the tangent to the graph is parallel to the x-axis. Hence the slope is zero and $\frac{dy}{dx} = 0$ for both these values.

From both explanations we conclude that, in general,* $\frac{dy}{dx} = 0$ *for values of x which make $y = f(x)$ a maximum or a minimum.* Values of x for which $\frac{dy}{dx} = 0$ are called *critical values.*

74. Tests for Maxima and Minima. The conclusion we reached at the end of the previous section is not sufficient for our purposes for two reasons: (1) there is nothing to show whether the value of $y = f(x)$ corresponding to a critical value $x = a$ is a maximum or a minimum; (2) sometimes, as the adjoining figure shows, there are critical values for which the function has neither a maximum nor a minimum value.

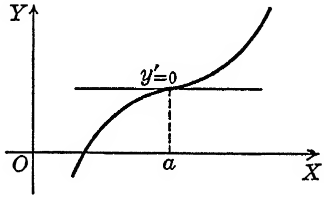


FIG. 69

Let us now consider an example. Let $y = x^3 - 12x$, whence $\frac{dy}{dx} = 3x^2 - 12$. Since the derivative must vanish at either a maximum or a minimum point, we shall find all such points by setting $\frac{dy}{dx} = 0$ and solving for x. If $3x^2 - 12 = 0$, then $x = \pm 2$, and these are the only values of x for which y can have either a maximum or a minimum value. Each

* There are some examples for which neither explanation is valid. For example, $y = x^{\frac{2}{3}}$ has a minimum value for $x = 0$. There is a sharp point (cusp) on the curve at $x = 0$, and the tangent is perpendicular to the x-axis. The derivative does not exist when $x = 0$. Functions of such exceptional character are not included in our consideration.

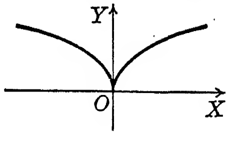


FIG. 70

of these two values of x must be tested separately, to see whether the corresponding value of y is a maximum or a minimum. Two tests will be given, either of which will serve as a criterion for maxima and minima.

First Test: by Computing Values of the Function. When $x = 2$, $y = -16$. For a value of x a little less than 2, say 1.5, $y = -14.625$; for a value of x a little greater than 2, say 2.5, $y = -14.375$. Since the value of y for $x = 2$ is less than either of the last two, we conclude that $x = 2$ gives a minimum value $y = -16$.

Second Test: by Investigating the Sign of $\frac{dy}{dx}$. Write the derivative in factor form:

$$\frac{dy}{dx} = 3(x+2)(x-2).$$

For a value of x a little less than 2, $\frac{dy}{dx} = 3(+)(-) = -$, which shows that y is decreasing. For a value of x a little greater than 2, $\frac{dy}{dx} = 3(+)(+) = +$, which shows that y is increasing. Hence $x = 2$ gives a minimum value, since y stops decreasing and begins to increase. (See also the illustrative examples in § 71.)

By either of these methods it may be shown that $x = -2$ gives a maximum value $y = 16$.

Summarizing, we have the following method:

General Method for Determining Maximum and Minimum Values of a Function $y = f(x)$

I. *Find* $\frac{dy}{dx} = f'(x)$ *in terms of* x.

II. *Set* $f'(x) = 0$ *and solve the equation.* This gives all the critical values of x.

III. *Examine each critical value by one of the tests on the following page.*

First Test. Let $x = c$ be a critical value. Find the three values of y corresponding (1) *to $x =$ some value slightly less than c;* (2) *to $x = c$;* (3) *to $x =$ some value slightly greater than c. If the value of y corresponding to $x = c$ is algebraically greater (less) than those on either side of c, then $y = f(c)$ is a maximum (minimum).*

Second Test. Let $x = c$ be a critical value. If $\frac{dy}{dx}$ is positive (negative) for a value of x a little less than c and negative (positive) for a value of x a little greater than c, then $y = f(c)$ is a maximum (minimum).

75. Examples. The general directions given at the close of the previous section suffice for problems in which the equation of the function is given. Sometimes, however, the equation is not given, and we are required to determine a maximum or minimum value of a quantity which satisfies certain specified conditions. In this kind of problem we must first express the quantity as a function of some *one* variable, and then proceed as above. (See Examples 3 and 4 below.) Furthermore, in many problems it is evident from the nature of the problem whether the critical value gives a maximum or a minimum, and in these cases the testing may be dispensed with.

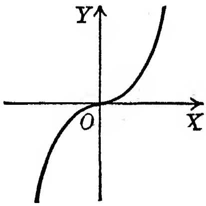


FIG. 71

EXAMPLE 1. Examine $y = x^3$ for maximum and minimum values.

Solution. Since $\frac{dy}{dx} = 3x^2$, the only critical value is $x = 0$. For both negative and positive values of x, $\frac{dy}{dx}$ is positive. Hence the function is always increasing (except at $x = 0$), and there is neither a maximum nor a minimum value.

EXAMPLE 2. Examine $y = x + \frac{1}{x}$ for maximum and minimum values.

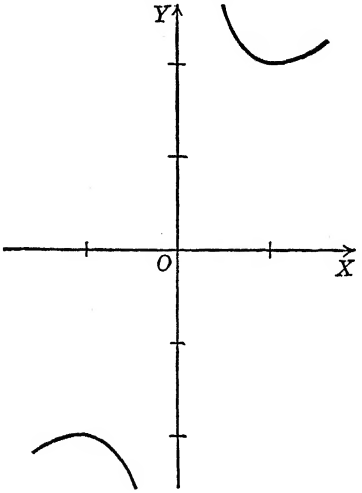


FIG. 72

Solution. Differentiating,

$$\frac{dy}{dx} = 1 - \frac{1}{x^2}.$$

Setting $\frac{dy}{dx} = 0$ and solving for x, we get $x = \pm 1$.

x	y	x	y
$-\frac{3}{2}$	$-2\frac{1}{6}$	$\frac{1}{2}$	$2\frac{1}{2}$
-1	-2	1	2
$-\frac{1}{2}$	$-2\frac{1}{2}$	$\frac{3}{2}$	$2\frac{1}{6}$

Applying the first test, we calculate the values shown in the tables. The first table shows that $x = -1$ gives a maximum value $y = -2$. The second table shows that $x = 1$ gives a minimum value $y = 2$. For $x = 0$, y becomes infinite.

If the second test is preferred, the derivative should be written as a fraction and then factored:

$$\frac{dy}{dx} = \frac{x^2 - 1}{x^2} = \frac{(x+1)(x-1)}{x^2}.$$

We now have the following results:

For $x < 1$, $\qquad y' = \frac{(+)(-)}{+} = -.$

For $x > 1$, $\qquad y' = \frac{(+)(+)}{+} = +.$

Hence y has a minimum for $x = 1$.

EXAMPLE 3. Determine the area of the largest rectangle which can be cut from a circular piece of cardboard 10 in. in diameter.

Obviously, the center of the largest rectangle must be at the center of the circle. For any rectangle the first cut, as A_1B_1 or A_2B_2 or A_3B_3, can be made at any place desired. If it is made near the center (A_1B_1), the rectangle will have a small area. If made at A_2B_2, the area will be larger. If made near the circumference (A_3B_3), the area is again small. Apparently there will be one rectangle having a larger area than any of the others, and this fact is assumed.

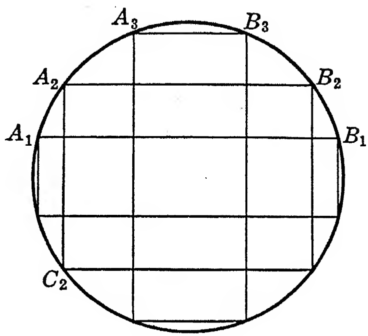


FIG. 73

Solution. The first step is to express the area A as a function of some *one* independent variable. Let x and y represent the lengths of the two sides of the rectangle. Then

$$A = xy \tag{1}$$

and

$$x^2 + y^2 = 100. \tag{2}$$

Substituting in the first equation the value of y from the second, we have

$$A = x\sqrt{100 - x^2}.$$

Differentiating,

$$\frac{dA}{dx} = \frac{-x^2}{\sqrt{100 - x^2}} + \sqrt{100 - x^2}.$$

Setting the derivative equal to zero, we get

$$\frac{-x^2}{\sqrt{100 - x^2}} + \sqrt{100 - x^2} = 0;$$

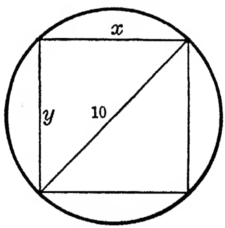


FIG. 74

whence

$$-x^2 + 100 - x^2 = 0$$

and

$$x = \sqrt{50}.$$

From equation (2), $y = \sqrt{50}.$

Hence the largest rectangle is a square of area 50 sq. in.

EXAMPLE 4. Find the altitude of the largest right circular cone which can be inscribed in a sphere of radius r.

Solution. The adjoining figure represents a vertical section. Let x be the radius of the cone and let y be its height. The volume V is to be a maximum.

By geometry,

$$V = \tfrac{1}{3}\pi x^2 y.$$

But V must be expressed as a function of *one* variable only. From the figure we see that

$$x^2 = r^2 - OD^2 = r^2 - (y-r)^2 = 2\,ry - y^2.$$

Substituting this above, we have

$$V = \frac{\pi}{3}(2\,ry^2 - y^3).$$

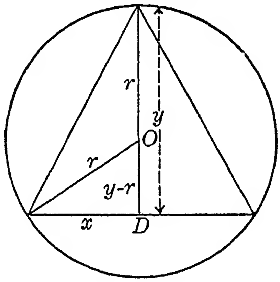


FIG. 75

Differentiating, and setting the derivative equal to 0,

$$\frac{dV}{dy} = \frac{\pi}{3}(4\,ry - 3\,y^2) = 0.$$

Solving, $$y = 0 \quad \text{or} \quad \frac{4\,r}{3}.$$

It is obvious from the conditions of the problem that there must be a maximum value of V and that the value $y = 0$ is meaningless. Hence we have the maximum cone when $y = \frac{4}{3}r$.

PROBLEMS

1. Examine the following functions for maxima and minima. Draw the graph in each case.

a. $y = x^2 - 5\,x + 3.$

b. $y = 1 + 7x - 2\,x^2.$

c. $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2\,x + 2.$

d. $y + x^3 + 12\,x^2 + 45\,x + 52 = 0.$

e. $6\,y = 2\,x^3 - 3\,x^2 - 36\,x.$

f. $y = 2\,x^3 - 3\,x^2 + 6\,x - 2.$

g. $y = x^4 - 2\,x^2.$

h. $15\,y = 3\,x^5 - 25\,x^3 + 60x.$

i. $y = x^4 - 4\,x^3 + 4\,x^2 - 4.$

j. $y = 6\,x^2 - x^4.$

k. $y = x^2 + \dfrac{16}{x}.$

l. $y = x^2 + \dfrac{1}{x^2}.$

m. $y = x^3 + \dfrac{3}{x}.$

n. $y = \dfrac{6\,x}{x^2 + 1}.$

2 Find two numbers whose sum is 20, such that

a. The sum of their squares is a minimum.

b. The sum of their cubes is a minimum.

c. Their product is a maximum.

d. The difference between one and the reciprocal of the other is a maximum.

3 A rectangular field to contain 40 A. is to be fenced off along the bank of a straight river. If no fence is needed along the river, what must be the dimensions requiring the least amount of fencing? (1 A. = 160 sq. rd.)

4. Show that of all triangles inscribed in a circle of radius a the equilateral triangle has the greatest area.

5. The legs of an isosceles triangle are each 20 in. long. Find the length of the base if the area is a maximum.

6. A trough is to be made of a long rectangular piece of tin by bending up two edges so as to give a rectangular cross section. If the width of the piece is 14 in., how deep should the trough be made in order that its carrying capacity may be a maximum?

7. Two upright poles, AB and CD, are 40 ft. apart. AB is 30 ft. high and CD is 20 ft. high. Find the distance AE if the length of the rope BED is a minimum.

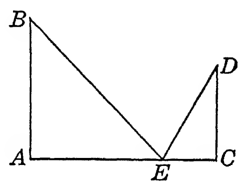


FIG. 76

A rectangular box is to be made from a sheet of tin 16 in. by 20 in. by cutting a square from each corner and turning up the sides. Find the edge of this square which makes the volume a maximum.

9. A rectangular box with a square base and a cover is to be built to contain 800 cu. ft. If the cost per square foot for the bottom is 15¢, for the top 20¢, and for the sides 10¢, what are the dimensions for a minimum cost?

10. A sheet of paper for a poster is to contain 16 sq. ft. The margins at the top and the bottom are to be 6 in., and those on the sides 4 in. What are the dimensions if the printed area is to be a maximum?

11. A rectangular box with a square base and an open top is to be made. Find the volume of the largest box that can be made from 320 sq. ft. of material.

12. The strength of a rectangular beam varies as the product of the breadth and the square of the depth. Find the dimensions of the strongest beam that can be cut from a cylindrical log whose diameter is a.

13. The stiffness of a rectangular beam varies as the product of the breadth and the cube of the depth. Find the dimensions of the stiffest beam that can be cut from a cylindrical log whose radius is a.

14. Two vertices of a rectangle are on the diameter of a semicircle of radius a, and the other two vertices are on the arc. Find the dimensions of the rectangle if its area is a maximum.

15. Two roads intersect at right angles, and a spring is located in an adjoining field 10 rd. from one road and 5 rd. from the other. How should a straight path just passing the spring be laid out from one road to the other so as to cut off the least amount of land? How much land is cut off?

16. Rectangles are inscribed in a circle of radius a. Find the dimensions of the rectangle whose perimeter has an extreme value, and show whether it is a maximum or a minimum.

17. One base of an isosceles trapezoid is the diameter of a circle of radius a, and the ends of the other base lie on the circumference of the circle. Find the length of the other base if the area is a maximum.

18. Find the length of the largest rectangular parcel with square ends which can be sent by parcel post (see Example 1, § 66).

19. Find the length of the largest cylindrical parcel which can be sent by parcel post (see Example 1, § 66).

20. What should be the diameter of a tin can holding 1 qt. (58 cu. in.) and requiring the least amount of tin (a) if the can is open at the top, (b) if the can has a cover?

21. A vertical cylindrical water tank, open at the top, is to contain 15,000 gal. Find the diameter if the material used is a minimum. (1 cu. ft. = 7.5 gal.)

22. Find the volume of the largest cylinder which can be cut from a given right circular cone whose height is h and whose base has the radius r.

23. The slant height of a right circular cone is a given constant a. Find the altitude if the volume is a maximum.

24. Find the dimensions of the right circular cylinder of maximum volume which can be cut from a solid wooden sphere of diameter 16 in.

25. Find the dimensions of the largest inscribed rectangular parallelepiped with a square base which can be cut from a solid sphere of radius r.

26. An oil can is made in the shape of a cylinder surmounted by a cone. If the radius of the cone is three quarters the height find the most economical proportions.

27. If the cost per hour for fuel required to run a given steamer varies as the cube of its speed and is $40 per hour for a speed of 10 mi. per hour, and if other expenses amount to $200 per hour, find the most economical rate to run it a distance of 500 mi.

28. A railroad company agreed to run a special train for 50 passengers at a uniform fare of $10 each. In order to secure more passengers, the company agreed to deduct 10¢ from this uniform fare for each passenger in excess of the 50 (that is, if there were 60 passengers, the fare would be $9 each). What number of passengers would give the company maximum gross receipts?

29. Find the area of the largest rectangle which can be inscribed in the ellipse whose equation is $x^2 + 4y^2 = 16$.

30. Find the dimensions of the largest rectangle which can be inscribed in the ellipse whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

31. Find the area of the largest rectangle which can be drawn with its base on the x-axis and with two vertices on the witch whose equation is $y = \dfrac{8a^3}{x^2 + 4a^2}$.

32. On the circle whose equation is $x^2 + y^2 = 100$ find the coördinates of the point which is nearest to the point $(12, 16)$. Is there any other way of solving this problem?

33. What point on the curve $4y = x^2$ is nearest to the point $(0, 4)$?

34. Show that the distance from the origin to the line whose equation is $\dfrac{x}{a} + \dfrac{y}{b} = 1$ is $\dfrac{ab}{\sqrt{a^2 + b^2}}$ by finding the point on the line whose distance from the origin is a minimum.

35. Find the shortest distance from the point $(10, 11)$ to the line whose equation is $3x + 4y = 24$ by finding the point on the line whose distance from $(10, 11)$ is a minimum.

36. The x-axis and y-axis are joined by lines which are tangent to the circle whose equation is $x^2 + y^2 = r^2$, where r is a constant. Find the length of the shortest of these lines.

37. A ditch is to be dug to connect the points A and B of the figure. The earth at the left of the line AD is soft, and the cost of digging the portion AC is \$10 per foot. The earth at the right of AD is hard, and the cost of digging the portion CB is \$20 per foot. Where should the turn C be made for a minimum cost?

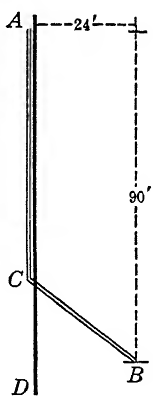


FIG. 77

38. A brick conduit, designed to accommodate underground cables, is to be built with a cross section in the form of a rectangle surmounted by a semicircle. Its carrying capacity (that is, the area of the cross section) is to be 24 sq. ft. If the cost of construction is assumed to be proportional to the perimeter of the cross section, find the width which will involve the least cost.

39. Show that $y = x^n$ has a minimum value if n is even, and that it has neither a maximum nor a minimum if n is odd.

40. Two towns are situated at distances of 3 mi. and 9 mi., respectively, from the shore of a lake, which is assumed to be a straight line. If the points on the shore nearest the towns are 5 mi. apart, at what point on the shore should a pumping-station, designed to supply both towns, be located so as to require the least amount of water mains? How many miles of mains are required?

41. Let $P(a, b)$ be a point in the first quadrant of a set of rectangular axes. Draw a line through P cutting the positive ends of the axes at A and B. Calculate the intercepts of this line on OX and OY in the following cases:

a. When the area OAB is a minimum.

b. When the length AB is a minimum.

c. When the sum of the intercepts is a minimum.

d. When the perpendicular distance from O to AB is a maximum.

42. An electric current flows about a coil of radius r, and exerts a force F on a small magnet the axis of which is on a line drawn through the center of the coil and perpendicular to its plane. This force is given by $F = \dfrac{x}{(r^2 + x^2)^{\frac{5}{2}}}$, where x is the distance to the magnet from the center of the coil. Show that F is a maximum for $x = \dfrac{r}{2}$.

CHAPTER VI

RATES AND DIFFERENTIALS

76. When one variable is expressed as a function of a second variable, the relation may be used to study the effect upon the function of a variation in the independent variable. We may draw the graph, taking the values of the independent variable as abscissas and those of the function as ordinates, and apply the geometric results of previous chapters. These show that the function is increasing when its derivative is positive, and is decreasing when its derivative is negative. We also note from the graph that at points where the curve is steep the function is increasing *rapidly* with respect to the independent variable, and vice versa; in other words, the rate of change of the function is measured by the slope of its graph. This is the fundamental idea upon which will be based the further study of the variation of a function corresponding to a given variation of the independent variable.

77. Constant Rates. A quantity may change at either a constant or a variable rate. We shall consider first the case of a constant rate. For this type of problem the processes of arithmetic and algebra are sufficient.

For example, suppose that oil is being pumped at a constant rate into a tank containing 10 gal. at 10.02 A.M. and 50 gal. at 10.12 A.M. We find easily that the contents are increasing at the rate of 40 gal. in 10 min., or 4 gal. per minute, and conclude that in the next 5 min. $5 \times 4 = 20$ more gallons will be added, in the next 10 min. 40 more, etc.

The foregoing is an example of a time rate, but many other kinds occur. As a second example, if a man walks along a railroad track with a 1 per cent grade, his altitude above sea level is a function of the distance walked, and is increasing at the rate of 52.8 ft. per mile.

The translation of the solution of the former example into the language of calculus will be instructive. The volume (V) of oil in the tank is evidently a function of the time (t), which we may conveniently measure from 10 o'clock. When $t = 2$ min., $V = 10$ gal.; when $t = 12$, $V = 50$. By subtracting the corresponding values of t and V we find that the respective increments are $\Delta t = 10$ and $\Delta V = 40$; that is, as t increases 10 min. V increases 40 gal. Hence the rate of change of V is $\dfrac{\Delta V}{\Delta t} = \dfrac{+40}{10} = +4$. Thus *the arithmetical method of finding a rate of change of one variable with respect to another is merely to divide the increment of the first variable by the corresponding increment of the second.*

78. Formula for a Quantity changing at a Constant Rate. If a variable quantity, y, changes with respect to another variable, x, at a known constant rate, and if we know a pair of corresponding values of x and y, we can at once write down the formula for y in terms of x.

Take as an example the oil tank considered in the previous section. There were 10 gal. in the tank at 10.02 o'clock, and this quantity was increasing at the constant rate of 4 gal. per minute. Therefore at any time t there will have elapsed $t - 2$ min. since 10.02, and the given value of V will have increased by $4(t-2)$ gal. Hence, at any time, $V = 10 + 4(t-2) = 2 + 4t$.*

*The same result is obtained if we use the fact that $V = 50$ when $t = 12$. Naturally, the formula is meaningless for $t < -\frac{1}{2}$, since then V would be negative.

As this equation is of the first degree, its graph is a straight line of slope 4. Thus we see that in this case the rate of change of V with respect to t is the slope* of its graph. (See also Example 2, § 66.)

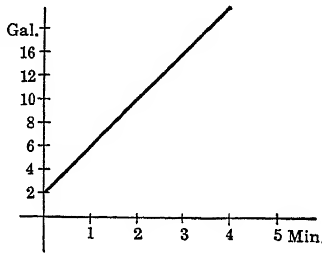


FIG. 78

In general, *if $y = b + mx$ is any linear function of x, the rate of change of y with respect to x is the slope of its graph.* For if we give x an increment Δx, we have successively

$$y + \Delta y = b + mx + m\Delta x$$

and $$\Delta y = m\Delta x.$$

But the rate is found by dividing Δy by Δx, giving

$$\frac{\Delta y}{\Delta x} = m, \text{ the slope of the line.}$$

(If the rate of change is negative, the function is decreasing.)

Conversely, *if y changes with respect to x at a constant rate m, and if $y = b$ when $x = 0$, then $y = b + mx$.*

Notation. For the sake of compactness the rate unit is usually written as a fraction. Thus a rate of 4 gal. per minute is written as 4 gal./min. or as $4 \frac{\text{gal.}}{\text{min.}}$. Similarly, a speed of 44 ft. per second is written as 44 ft./sec. or as $44 \frac{\text{ft.}}{\text{sec.}}$. This notation will be used in the future.

In all problems involving rates the units should be clearly stated. Thus in Problem 3 below it is insufficient to say that the rate of change is 0.019. The correct form is 0.019 cm. per degree.

*The slope is equal to the tangent of the inclination only when the units are the same on both axes. When the units are different, the slope of the line joining two points is defined as $m = \frac{y_1 - y_2}{x_1 - x_2}$.

PROBLEMS

1. In experiments on the temperatures at various depths in an artesian well the temperature ($T°$) was found to be connected with the depth (d ft.) by the equation $T = 52 + 0.017\,d$. What was the rate of change in the temperature with respect to the depth? Draw the graph of the equation.

2. In experiments with a pulley block the pull p (lb.) required to lift a weight w (lb.) was found to be $p = 0.03\,w + 0.5$. How much does the pull increase per hundredweight increase in w?

3. A metal bar increases in length at a constant rate if heated. Suppose that at 20° C. its length is 1000 cm. and that at 60° C. its length is 1000.76 cm. Find its increase in length per degree increase in temperature. Express the length as a function of the temperature.

4. The extension of the spring of a balance is proportional to the weight attached to it. The length of the spring is 4.25 in. under a weight of 2 lb., and 5 in. under a weight of 8 lb. Express the length of the spring as a function of the weight. What is the natural length of the spring?

5. Fahrenheit temperature (F) is connected with centigrade temperature (C) by the relation $F = \frac{9}{5}\,C + 32$. Draw the graph of this function on a large enough scale to make it possible to read off corresponding values of F and C correct to one degree.

6. At a height of 110 m. above sea level the barometer reads 750 mm.; at a height of 770 m. it reads 695 mm. If it is assumed that the barometric pressure decreases constantly as the height increases, find the rate at which it decreases with respect to the height. What will be the pressure at a height of 850 m.? Express the pressure as a function of the height.

7. The boiling-point of water decreases as the altitude above sea level increases. At an altitude of 500 ft. the boiling-point is 211° and at 2500 ft. it is 207°. Find the rate of change, assuming that it is constant. What will be the boiling-point at an altitude of 12,000 ft.?

8. Water is flowing out of a cylindrical tank of radius 5 ft. at the constant rate of 20 gal./sec. How fast is the surface falling? If the depth was 15 ft. when $t = 0$, express the depth as a function of t and draw its graph. (1 cu. ft. = 7.5 gal.)

9. Oil is being pumped into a cylindrical tank of diameter 4 ft. at the constant rate of 5 gal./min. How fast is the surface of the oil in the tank rising?

10. If $3x + 5y = 8$, at what rate does y change with respect to x? Illustrate by a graph.

11. Show that if $y = 4 - 8x - x^2$, y does not change with respect to x at a constant rate.

12. The velocity of a moving point is given by the formula $v = 50 - 6t$, where v is measured in feet per second and t is measured in seconds. At what rate is v changing?

13. Show that the circumference of a circle changes at a constant rate with respect to the radius.

14. If a variable point P moves from (2, 8) to (6, -2) along a straight line, at what rate is the ordinate changing with respect to the abscissa?

79. Variable Rates. If, on the other hand, the pump (§ 78) was working at a variable rate, the result just obtained cannot be used as before. We cannot say that in the next 5 min. the contents of the tank will increase by 20 gal., nor can we say that exactly 4 gal. were added in any 1 min. of the 10 under consideration. What we do commonly say is that the pump was working "on the average" at the rate of 4 gal./min. between 10.02 and 10.12.

In general we have this definition: *If $y = f(x)$, the quotient $\dfrac{\Delta y}{\Delta x}$ is the average rate of change of y with respect to x as x increases from the value x to $x + \Delta x$.* When the average rate is constant, it is of course equal to the change in y corresponding to each unit increment of x.

EXAMPLE. What is the average rate at which area is added to a square if the length of its side increases from 5 to 9 in.?

Solution. Let A be the area and x the length of the side of the square. From geometry we have

$$A = x^2.$$

Give x an increment Δx, and we have

$$A + \Delta A = x^2 + 2x\Delta x + (\Delta x)^2.$$

Hence $$\Delta A = 2x\Delta x + (\Delta x)^2,$$

and the average rate of increase for any values of x and Δx is

$$\frac{\Delta A}{\Delta x} = 2x + \Delta x.$$

For the case given, $$x = 5$$

and $$\Delta x = 9 - 5 = 4.$$

$$\frac{\Delta A}{\Delta x} = 10 + 4 = 14.$$

Hence the required average rate is 14 sq. in. per inch increase in the length of the side. Note that the average rate is a function of both x and Δx.

x	A
0	0
5	25
9	81
10	100
15	225
20	400

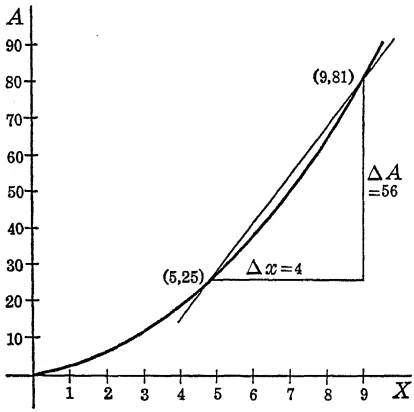


FIG. 79

In this case the average rate appears in the graph as the slope of the secant (if allowance is made for the different scales used for A and x) joining the points (5, 25) and (9, 81).

Moreover, successive values of A cannot be computed by multiplying this rate by the increments of x and adding the results to the value of A for $x = 5$, as can be done when the rate is constant. Fair approximations are obtained when Δx does not differ much from 4; but as Δx increases, the results become more useless; in fact, as is easily seen from the figure, such a process gives merely the ordinates of points on the secant line.

80. Instantaneous Rates. In the example of the preceding section it was found that as x increases from 5 to 9 the area increases at the average rate of 14 sq. in. per inch increase in the length of the side. As x increases from 5 to 8 we find, by substituting $x = 5$ and $\Delta x = 3$ in the expression for $\frac{\Delta A}{\Delta x}$, that the average rate of increase in the area is 13 sq. in./in. Similarly, as x increases from 5 to 7 ($\Delta x = 2$), the average rate of increase of the area is 12 sq. in./in. As Δx is taken smaller and smaller and made to approach zero as a limit, the average rate of increase of the area approaches the limiting value 10 sq. in./in., which is the *instantaneous rate of change* of the area with respect to the side at the instant when the latter is 5 in.

In general we have the following definition:

The instantaneous rate of change of y with respect to x is the limit of the average rate of change in the interval Δx as Δx approaches zero.

We can now prove the following important theorem:

Theorem. *If $y = f(x)$, the instantaneous rate of change of y with respect to x for $x = a$ is the value of $\frac{dy}{dx}$ for $x = a$.*

Proof. For any values of x and for $\Delta x \neq 0$ the average rate of change of y is $\dfrac{\Delta y}{\Delta x}$.

The instantaneous rate is the limit of the average rate as $\Delta x \to 0$, by definition. Hence

$$\text{Instantaneous rate} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \frac{dy}{dx}. \qquad \text{By definition of derivative}$$

Corollary. The rate of change of any function with respect to its variable is equal to the slope of its graph at the given point.

EXAMPLE 1. If $y = 3x - \dfrac{x^3}{6}$, at what rate is y changing with respect to x when $x = 3$?

Solution. Differentiating,

$$\frac{dy}{dx} = 3 - \frac{x^2}{2}.$$

When $x = 3$, $\qquad \dfrac{dy}{dx} = 3 - 4.5 = -1.5.$

Hence y is decreasing at the rate of 1.5 units per unit increase in x. Note that this is the same as the slope of the graph at the point where $x = 3$. In fact, the slope may be regarded as merely the rate of change of the ordinate with respect to the abscissa.

EXAMPLE 2. At noon a ship bound north is 60 mi. south of another ship, which is bound east. If the first ship is sailing at the rate of 15 mi./hr., and the second ship at the rate of 10 mi./hr., how fast is the distance between them changing at 2 o'clock? at 3 o'clock?

Solution. Let A and B be the first positions of the ships and C and D their positions after t hours. Then $BD = 10t$ and $CB = 60 - 15t$. Let z be the distance between them. Then we have at once

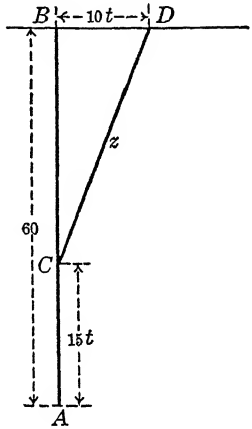


FIG. 80

$$z = \sqrt{CB^2 + BD^2}$$
$$= \sqrt{(60 - 15t)^2 + 100t^2}$$
$$= \sqrt{3600 - 1800t + 325t^2}.$$

To find the rate at which z is changing, differentiate:

$$\frac{dz}{dt} = \frac{325t - 900}{\sqrt{3600 - 1800t + 325t^2}}.$$

At 2 o'clock $t = 2$ and

$$\frac{dz}{dt} = \frac{-250}{\sqrt{1300}} = -6.9;$$

that is, the ships are approaching each other at the rate of 6.9 mi./hr. When $t = 3$, $\frac{dz}{dt} = \sqrt{5} = 2.2$; that is, the ships are separating at the rate of 2.2 mi./hr.

81. Rectilinear Velocity and Acceleration. Several kinds of rates occur so often that names have been given to them. Among these are *velocity* and *acceleration*.

FIG. 81

The velocity of a point moving along a straight line is merely the rate at which its distance from some fixed point is changing with respect to the time. Thus, if s represents the distance of P from the fixed point O at any time t, the velocity of P is given by the formula

$$v = \frac{ds}{dt}.$$

Since a function is increasing (algebraically) if its derivative is positive, and decreasing if its derivative is negative, it follows that the point is moving in the positive direction along the line if the velocity is positive and that it is moving in a negative direction if the velocity is negative. The *speed* is the numerical value of the velocity.

The acceleration of a point moving along a straight line is the rate at which its velocity is changing with respect to the time. Hence, if a denotes acceleration and v denotes velocity, we have at once

$$a = \frac{dv}{dt}.$$

Since v itself is the derivative of s with respect to t, this makes a the second derivative (see § 62) of s with respect to t; that is,

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}.$$

EXAMPLE. A point moves in a straight line directed vertically upward according to the law $s = 96\,t - 16\,t^2$. Find (*a*) its velocity and acceleration after 4 sec., (*b*) how high it will rise, (*c*) how far it will move in the fifth second.

FIG. 82

Solution. At any time,

$$v = \frac{ds}{dt} = 96 - 32\,t$$

and

$$a = \frac{dv}{dt} = -32.$$

(*a*) When $t=4, v=96-32\,t=96-128=-32$ ft./sec., indicating that the point is coming *down* at the rate of 32 ft./sec. Since the acceleration is negative, the velocity is always decreasing at the rate of 32 ft. per second per second. The speed is decreasing when the point is moving upward and is increasing when the point is moving downward.

(*b*) The body will cease to rise when $v = 0$; therefore

$$96 - 32\,t = 0, \quad \text{or} \quad t = 3 \text{ sec.}$$

Its height will be the value of s for $t = 3$; that is,

$$s = 96 \cdot 3 - 16 \cdot 3^2 = 288 - 144 = 144 \text{ ft.}$$

(*c*) To find the distance moved in the fifth second, find the values of s for $t = 4$ and $t = 5$; by substitution in the given formula these are seen to be 128 and 80 respectively. As the latter height is less than the former, it is seen that during the fifth second the body came down $128 - 80 = 48$ ft. Compare this result with the velocity at the beginning of the fifth second, which was found in (*a*) to be -32 ft./sec.

NOTE. The acceleration of a body falling freely in a vacuum is called the acceleration of gravity and is denoted by g. This number varies slightly in different parts of the earth, but is approximately 32 ft. per second per second (sometimes written 32 ft./sec.^{2}). That is, if the positive direction is taken as downward, the velocity of a body moving under the given conditions increases by 32 ft./sec. each second. If the positive direction is taken as upward, the effect is to decrease the velocity, and the acceleration is $-g$, or -32. The same acceleration is frequently used in studying the motion of bodies falling in the air, and fairly good approximations may be obtained provided the velocities are not excessive and the falling body is heavy in proportion to its volume.

PROBLEMS

1. If $y = x^2 + \dfrac{4}{x^2}$, find (*a*) the average rate of change of y with respect to x as x increases from 2 to 4, (*b*) the instantaneous rate when $x = 2$, (*c*) the actual change in y when x changes from 2 to 2.5.

2. If $y = x^3 - 4\,x^2 + 6$, find the rate at which y is changing with respect to x when $x = 1$. Using this rate, approximate the change in y when x changes from 1 to 1.2. Why is your result not exact?

3. Write the equation and draw the graph of a function which never decreases with respect to its variable.

4. If $xy^2 = 36$, what is the rate of change of y with respect to x when $y = 3$? Illustrate your result by a graph.

5. If $s = a + bt$, where a and b are constants, show that the acceleration is 0.

6. If $s = a + bt + ct^2$, where a, b, and c are constants, show that the acceleration is constant.

7. A particle falls according to the law $s = \frac{1}{2}gt^2$. Find its average velocity during the first 5 sec. What is the speed at the end of 4 sec.? How far does it fall during the next 0.1 sec.?

8. If the distance s is measured in feet and t in seconds, find the velocity at the end of 2 sec. when (*a*) $s = (1 - t^2)^{-1}$, (*b*) $s = \sqrt{t} + 1$.

9. In Problem 8 find the acceleration at the end of 2 sec.

10. How fast is the reciprocal of x changing with respect to x when $x = 0.1$? when $x = 10$?

11. Find the rate of increase in the volume of a sphere with respect to the radius when the radius is 14 in. From this result approximate the change in the volume if the radius increases from 14 to 14.3 in.

12. At a certain instant a ship bound north is 6 mi. west of another ship, which is bound east. If the first ship is sailing at the rate of 15 mi./hr. and the second ship at the rate of 12 mi./hr., how fast are they separating at the end of one hour?

13. Two railroad tracks intersect at right angles. At noon there is a train on each track approaching the crossing at 40 mi./hr., one being 100 mi. and the other 200 mi. distant. Find (*a*) how fast they are approaching each other at 1 o'clock, (*b*) when they will be nearest together, (*c*) what will be their minimum distance apart.

14. As a man walks across a bridge at the rate of 4 ft./sec. a boat passes directly beneath him going downstream at 10 ft./sec. If the bridge is 30 ft. above the water, how fast are the man and the boat separating 3 sec. later?

15. It is found by experiment that the volume of water which at 4° C. has unit volume is given by the equation $V = 1 + a(t - 4)^2$, where t denotes the temperature and $a = 0.00000838$. Find the rate at which the volume is changing when $t = 0°$ and when $t = 20°$.

16. The time (t seconds) of a complete oscillation of a pendulum of length l inches is given by the formula $t = 0.324\sqrt{l}$. Find the rate of change of the time with respect to the length of the pendulum when $l = 9$ in. By means of this result approximate the change in t due to a change in l from 9 in. to 9.2 in.

17. If there is no transfer of heat, the pressure and volume of compressed air are connected by the relation $pv^{1.41} = C$, where C is a constant. If $p = 20$ lb./sq. in. when $v = 800$ cu. in., find the rate of change in p with respect to v when $v = 800$ cu. in., and approximate the resulting change in p if v is decreased 10 cu. in.

82. Related Rates. Frequently the time rate of change of one variable is known, and it is desired to find the time rate of change of a second variable which is related to the first. Such problems are easily solved by differentiating the equation connecting the variables implicitly with respect to the time and substituting the given values of the variables.

EXAMPLE 1. A barge whose deck is 5 ft. below the level of a dock is drawn up to it by means of a cable running over a pulley at the edge of the dock, the cable being hauled in at the rate of 4 ft./min. How fast is the barge moving when it is 12 ft. away from the dock?

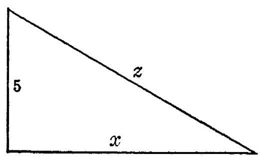


FIG. 83

Solution. Let z denote the length of the cable from the pulley to the barge and let x denote the distance of the barge from the dock *at any time.* We are given $\frac{dz}{dt} = -4$ and are required to find $\frac{dx}{dt}$ when $x = 12$.

The equation connecting x and z is seen from the figure to be

$$z^2 = x^2 + 25. \tag{1}$$

Differentiating with respect to t,

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt}. \tag{2}$$

We are given the values of $\frac{dz}{dt}$ and x; we find by substitution in equation (1) that $z = 13$. Substituting these values in equation (2), we obtain

$$2(13)(-4) = 2(12)\frac{dx}{dt},$$

or

$$\frac{dx}{dt} = -\frac{13}{3} = -4\tfrac{1}{3}.$$

That is, the barge is moving toward the dock at the rate of $4\frac{1}{3}$ ft./min. Note that a positive result would be erroneous, for that would indicate that x is increasing, not decreasing.

The above problem is typical, and in general the same procedure should be followed. The various steps are

I. *Determine the variables involved. These always include those for which the rates of change are given or desired.*

II. *Set up the equation connecting the variables.*

III. *Differentiate the equation implicitly with respect to t.*

IV. *Substitute the given values of the variables in the two equations and solve for the unknowns.*

Example 2. The radius of the base of a right circular cone is decreasing at the rate of 4 in./min., and the height is increasing at the rate of 6 in./min. At what rate is the volume changing when the height is 12 in. and the radius is 6 in.?

Solution. Let V, r, and h denote the volume, radius, and height, respectively, of the cone. We are given $\frac{dr}{dt} = -4$ and $\frac{dh}{dt} = 6$; we are to find $\frac{dV}{dt}$ when $h = 12$ and $r = 6$.

The formula for the volume of a cone is $V = \frac{\pi r^2 h}{3}$.

Differentiating this with respect to t, we have

$$\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt} + \frac{2}{3}\pi r h \frac{dr}{dt}.$$

Substituting the given values of the variables, we obtain

$$\begin{aligned}\frac{dV}{dt} &= \frac{\pi}{3}(36)(6) + \frac{2}{3}\pi(6)(12)(-4)\\ &= 72\pi - 192\pi = -120\pi = -377,\end{aligned}$$

indicating that the volume is decreasing at the rate of about 377 cu. in./min.

EXAMPLE 3. A point is moving along the ellipse $x^2 + 4y^2 = 16$. Find the points where x is increasing at the same rate as y is decreasing.

Solution. Differentiating the given equation with respect to t, we have

$$2x\frac{dx}{dt} + 8y\frac{dy}{dt} = 0.$$

We are given

$$\frac{dx}{dt} = -\frac{dy}{dt}.$$

Substituting this in the previous equation, we obtain

$$-2x\frac{dy}{dt} + 8y\frac{dy}{dt} = 0.$$

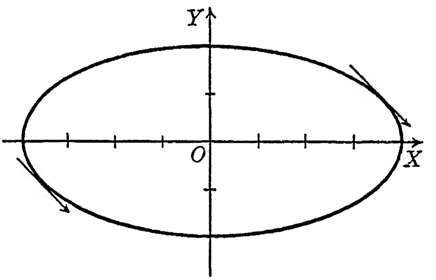


FIG. 84

Canceling $\frac{dy}{dt}$ and solving for x, we get $x = 4y$. Solving this simultaneously with the equation of the ellipse, we obtain

$$y = \pm\frac{2}{\sqrt{5}} = \pm 0.9$$

and

$$x = \pm\frac{8}{\sqrt{5}} = \pm 3.6.$$

Hence the required points are (3.6, 0.9) and (− 3.6, − 0.9). The figure confirms the pairing of the values.

PROBLEMS

1. A ladder 24 ft. long leans against a vertical wall. If the lower end is being moved away from the wall at the rate of 3 ft./sec., how fast is the top descending when the lower end is 8 ft. from the wall?

2. In Problem 1 find when the lower and the upper ends are moving at the same rate.

3. A conical funnel is 14 in. across the top and 12 in. deep. A liquid is flowing in at the rate of 60 cu. in./sec., and flowing out at the rate of 40 cu. in./sec. Find how fast the surface of the liquid is rising when it is 6 in. deep.

4. A man 6 ft. tall walks away from an arc light 15 ft. high at the rate of 3 mi./hr. (*a*) How fast is the farther end of his shadow moving? (*b*) How fast is his shadow lengthening?

5. A kite is 80 ft. high, with 100 ft. of cord out. If the kite starts moving away horizontally at the rate of 5 mi./hr., how fast is the cord being paid out?

6. A boat is fastened to a rope which is wound about a windlass 20 ft. above the level at which the rope is attached to the boat. The boat is drifting away at the rate of 8 ft./sec. How fast is it unwinding the rope when 30 ft. from the point directly under the windlass?

7. The volume of a sphere is increasing at the rate of 16 cu. in./sec. How fast is the radius increasing when it is 6 in.?

8. Find how fast the surface is increasing in Problem 7.

9. Sand is being poured on the ground from an elevated pipe and forms a pile which has always the shape of a right circular cone whose height is equal to the radius of the base. If the sand falls at the rate of 6 cu. ft./min., how fast is the height of the pile increasing when the height is 5 ft.?

10. The radius of a cone is decreasing at 2 in./min., and the altitude is increasing at the rate of 3 in./min. When the radius

is 18 in. and the altitude is 20 in., find (*a*) the rate at which the volume is changing, (*b*) the rate at which the curved surface is changing.

11. The diameter of a hemispherical bowl is 18 in. If the depth of the water in it is increasing at the rate of $\frac{1}{8}$ in./sec. when it is 8 in. deep, how fast is the water flowing in? (The volume of a segment of a sphere of radius r is $\pi h^2\left(r - \frac{h}{3}\right)$, where h is the height of the segment.)

12. If $y = x^2$, and x is increasing at the rate of $\frac{1}{2}$ unit/min. when $x = 2$, find (*a*) how fast y is changing, (*b*) how fast the slope of the graph is changing.

13. The path of a moving point is the curve $x^2 - 4y^2 = 36$. If x increases steadily at the rate of 2 units/min., find how fast y is changing at the point $(10, -4)$.

14. The velocity of a jet of liquid issuing from an orifice is given by the formula $v^2 = 2gh$, where h is the height of the liquid surface above the orifice. If h is decreasing at the rate of 3 in./min., find how fast the velocity of flow is changing when $h = 100$ ft.

15. If $y^2 = 2x$, and x is decreasing steadily at the rate of 0.25 units/min., find how fast the slope of the graph is changing at the point $(8, -4)$.

16. If $100y = 400 - x^2$, and y is increasing at the rate of 0.1 unit/min. when $x = 20$, find (*a*) how fast x is changing, (*b*) how fast the slope of the graph is changing.

17. A point moves along the parabola whose equation is $4y = x^2$ so that the abscissa increases at the constant rate of 2 units per second. At what rate is the distance between this point and the point $(0, 4)$ changing when $x = 2$ and when $x = \sqrt{8}$?

18. The velocity of a point moving along a straight line is given by $v^2 = a + \frac{2b}{s}$, where a and b are constants. Show that the acceleration is $-\frac{b}{s^2}$.

83. Relation between the Increment of a Function and its Rate of Change. If $y = f(t)$ changes at a constant rate, the amount of change in y during any time interval Δt may be found by multiplying the rate by Δt. Thus, if y is the height of a balloon which is rising at the constant rate of 6 ft./sec., (that is, $\frac{dy}{dt} = 6$) the height will increase by $6\,\Delta t$ feet in Δt seconds, or $\Delta y = 6\,\Delta t$. Similar remarks, of course, apply to other constant rates of change, and it is this use of rates which the student has most frequently met in arithmetic and algebra. This may be stated formally as follows:

If the rate of change of y with respect to x is constant, $\Delta y = \Delta x$ times the rate of change of y with respect to x.

When the rate of change is not constant, this equation is no longer true. But the product of the rate by Δx will give an approximation to Δy, which is increasingly accurate as Δx is made smaller. For example, consider the area of a square, which is given by the formula $A = x^2$. The rate of change of A with respect to x is $\frac{dA}{dx} = 2x$ for any value of x; and when $x = 25$, this becomes 50 sq. in./in. This would indicate that an addition of $\Delta x = 2$ in. to the length of the side of the square would increase the area by $\Delta A = 2 \times 50 = 100$ sq. in. But the actual increase in the area is $(27)^2 - (25)^2 = 104$ sq. in.,— a discrepancy of 4 sq. in. When $\Delta x = 0.5$ in. the use of the rate of change would make $\Delta A = 0.5 \times 50 = 25$ sq. in., while the true value of $\Delta A = (25.5)^2 - (25)^2 = 25.25$ sq. in. In this case the calculation of ΔA by means of the rate of change is in error only 0.25 sq. in. In ordinary speech we explain this by saying that the rate may be regarded as constant if the change in x is only small enough. This principle leads to the important notion of differentials.

84. Differentials Defined. *The differential of a function of a variable is the product of its derivative with respect to the variable by the increment of the variable.*

The differential of a function is denoted by the letter d prefixed to the symbol denoting the function. The above definition may then be stated thus: If $y=f(x)$, $dy=df(x)=f'(x)\Delta x=\frac{dy}{dx}\Delta x$. Evidently dy is a function both of x and of Δx.

If $f(x)=x$, $f'(x)=1$. Hence $dx=1\Delta x=\Delta x$; or the differential of the *independent* variable equals the increment of the independent variable. On this account it is customary to write the relation $dy=f'(x)\Delta x$ in the form $dy=f'(x)\,dx$, and this practice will be followed in the future.

The student must not, however, make the mistake of thinking that the differential and the increment of the dependent variable are identical. The difference is shown geometrically in the following section.

If we remember that $f'(x)$ is the rate at which $y=f(x)$ increases with respect to x, *it appears that dy is the approximation to the increment of y corresponding to Δx, which we obtain by assuming that Δx is small enough to permit us to regard y as increasing at a constant rate.*

85. Geometric Meaning of the Differential. In the following figures let P be any point on the graph of $y=f(x)$, let PS be the tangent at P, and let Q be a neighboring point on the curve.

Evidently $PM=\Delta x=dx$, and $MQ=\Delta y$. Moreover, angle $SPM=\alpha$.

Hence $$m=\tan\alpha=\frac{MS}{dx};$$
or, since $m=f'(x)$, $$MS=f'(x)\,dx.$$
But, by definition, $$dy=f'(x)\,dx.$$
Therefore $$dy=MS.$$

Thus we have shown that *if $P(x, y)$ is a point on the curve $y = f(x)$, then, for a particular value of x and an arbitrarily chosen value of the increment dx, Δy is the*

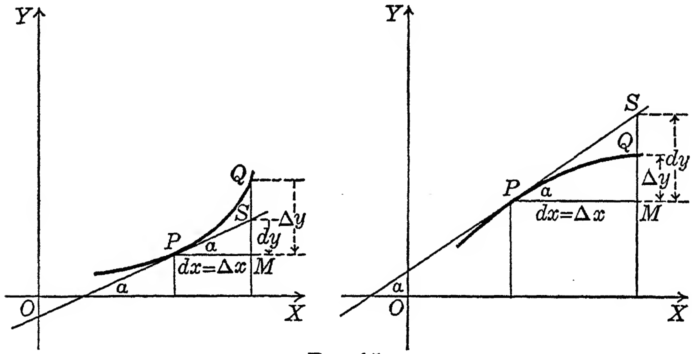


FIG. 85

corresponding increment of the ordinate drawn to the curve, and dy is the corresponding increment of the ordinate drawn to the tangent at P.

It is instructive to calculate the increment of y and the differential of y for a particular case. Let us take the curve whose equation is $4y = x^2$ and find the values of dy and Δy for $x = 1$ and $dx = 0.5$.

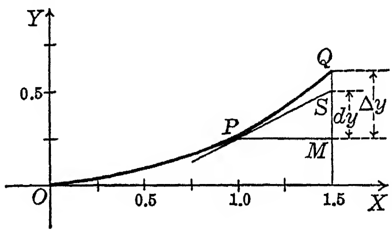


FIG. 86

Differentiating, we find that $\dfrac{dy}{dx} = \dfrac{x}{2}$, and hence $dy = \dfrac{x}{2}dx$. For $x = 1$ and $dx = 0.5$ this gives $dy = 0.25$.

When $x = 1$, $y = \dfrac{1}{4} = 0.25$; when $x = 1.5$, $y = \dfrac{2.25}{4} = 0.56$. Hence $\Delta y = 0.31$. Thus Δy differs from dy by 0.06.

It is also obvious from the figures that as Δx becomes smaller, dy and Δy become more nearly equal. To confirm this let us find dy and Δy when $\Delta x = 0.1$. As above, we obtain $\Delta y = 0.0525$, $dy = 0.05$, and $\Delta y - dy = 0.0025$ only.

86. How to find Differentials. The definition indicates the method of finding the differential of a given function; namely, to find the derivative, and to multiply it by the differential of the variable. Therefore, to every formula giving a derivative there corresponds one giving the differential. For example, $dc = 0$; $d(u^n) = nu^{n-1}du$; etc.

EXAMPLE 1. Find the differential of $x(2-3x)^2$.

Solution.
$$d[x(2-3x)^2] = \frac{d}{dx}[x(2-3x)^2]\,dx$$
$$= (2-3x)(2-9x)\,dx.$$

EXAMPLE 2. Find dy and Δy if $y = x^2$.

Solution.
$$dy = d(x^2) = \frac{d}{dx}(x^2)\,dx = 2\,x\,dx.$$
$$\Delta y = f(x+dx) - f(x) = (x+dx)^2 - x^2$$
$$= 2\,x\,dx + \overline{dx}^2.$$

Thus the difference between Δy and dy is $\overline{dx}^2$, a quantity which is comparatively negligible if dx is small.

When the function is implicitly expressed, it is usually simpler to get the differential of each term, regarding both variables as independent, and then to solve for dy in terms of dx.

EXAMPLE. Find dy, if $xy = x^2 - 1$.

Solution.
$$d(xy) = d(x^2) - d(1).$$
$$x\,dy + y\,dx = 2\,x\,dx - 0.$$
$$dy = \frac{2\,x\,dx - y\,dx}{x} = \frac{2\,x - y}{x}\,dx.$$

PROBLEMS

1. Find the differentials of each of the following functions:

a. $x(1-x^2)^3$.

b. $\dfrac{\sqrt{1+x}}{4x}$.

c. $\dfrac{1}{x}$.

d. $\dfrac{y-1}{y+1}$.

e. $t\sqrt{1+t}$.

f. $x\sqrt{a-bx}$.

g. $\dfrac{1}{t^2}$.

h. $\dfrac{4\pi r^3}{3}$.

2. Find dy in terms of x, y, and dx if

a. $\sqrt{x}+\sqrt{y}=\sqrt{a}$.

b. $x^3+y^3=3axy$.

c. $b^2x^2-a^2y^2=a^2b^2$.

d. $xy=6$.

e. $x^2y^3+4y=4$.

f. $xy+y^2+4x=0$.

3. Find algebraically dy and Δy if $y=x^3$. Also find their difference if $x=1$ and $dx=0.1$.

4. Find algebraically ds and Δs if $s=16t-8t^2$. Also find their difference if $t=3$ and $\Delta t=0.5$.

5. Show geometrically that, if y is a linear function of x, dy and Δy are equal.

6. In each of the following equations find dy for the values of x and dx given. In each case draw a small arc of the curve near the point named and mark dx, dy, and Δy.

a. $y=7-6x+x^2$, $x=2$, $dx=0.3$.

b. $y=\sqrt{x}$, $x=1$, $dx=0.4$.

c. $y=x^3-4x^2$, $x=3$, $dx=0.2$.

d. $x^2+y^2=25$, $x=4$, $y=-3$, $dx=0.3$.

7. Find algebraic expressions for dy and Δy if $y=x^2$. From these find the error in assuming $(x+dx)^2=x^2+d(x^2)$ if $dx=0.1$.

8. Show that $(x+dx)^3$ is approximately equal to x^3+3x^2dx if dx is small. Use this result to approximate $(10.2)^3$.

9. Show that $f(x+\Delta x)\doteq f(x)+df(x)$, approximately.

87. Approximation of Increments and Errors by Means of Differentials. The fact that for small values of dx the corresponding increment of $f(x)$ is closely approximated by $df(x)$ is of considerable utility in approximating quantities which can be expressed as increments of functions.

For example, let it be required to find the amount of material needed to make a hemispherical shell of inner radius 10 in. and of thickness $\frac{1}{8}$ in. This is exactly the difference of two hemispheres of radii 10 in. and 10.125 in. respectively. If we let V be the volume of a hemisphere of radius R, $V=\frac{2}{3}\pi R^3$, and the desired result is the value of ΔV when $R=10$ and $\Delta R=0.125$. To calculate this is troublesome, and, if an approximation only is desired, the more easily obtained value of dV will suffice, since the differential is approximately equal to the increment when the increment of the independent variable is small (see §§ 84, 85). Hence the amount required is approximately

$$dV=2\pi R^2 dR=2\pi(100)(0.125)=25\pi=78.5 \text{ cu. in.}$$

Note that $dV=2\pi R^2 dR$ is merely the product of the inner surface of the shell by the thickness.

Good judgment must be employed in using this method. If, in the example given above, dR were too large, the result would be worthless. In this particular case the exact value, ΔV, is 79.5 cu. in.

Example. Use differentials to approximate $\sqrt[3]{127}$ and $\sqrt[3]{123}$.

Solution. Since the number 127 differs but little from the perfect cube 125, we can get a good approximation by finding the value of the differential of $y=\sqrt[3]{x}$ for $x=125$ and $dx=2$. Differentiating,

$$dy=\tfrac{1}{3}x^{-\frac{2}{3}}dx\Big]_{x=125,\ dx=2}$$
$$=\tfrac{2}{75}=0.027.$$

Hence $\sqrt[3]{127}=y+dy=5+0.027=5.027.$

To find $\sqrt[3]{123}$ observe that $dx = -2$ and hence $dy = -0.027$. Therefore

$$\sqrt[3]{123} = 4.973.$$

As dx becomes larger the method of course becomes increasingly unreliable.

Errors. When the value of a function is obtained by calculation, it may happen that the value of the independent variable is not exactly known, as is always the case when it is obtained by measurement. It is then desirable to know approximately the error in the function due to the possible small error in the variable. This is equivalent to approximating the increment in the function corresponding to a small increment in the variable. Hence the differential of the function is the desired error, approximately.

EXAMPLE. The height and diameter of a cylinder are measured, and are found to be 6 in. and 4 in. respectively. Assuming that the first measurement can be found exactly, but that the diameter may be in error $\frac{1}{50}$ in., find the error in the calculated volume. Also find how accurately the diameter must be measured in order that the error in the calculated volume shall not exceed 0.1 cu. in.

Solution. Since $h = 6$ exactly, we have $V = 6\,\pi r^2$. The error in r is one half that in the diameter; that is, $dr = \pm\, 0.01$ in. Hence the possible error in V is

$$dV = 12\,\pi r\,dr = \pm\, 12\,\pi(2)(0.01) = \pm\, 0.24\,\pi = \pm\, 0.75 \text{ cu. in.}$$

To answer the second question, we have

$$dV = 12\,\pi R\,dR, \text{ where } R = 2 \text{ and } dV = \pm\, 0.1.$$

Hence $\pm\, 0.1 = 24\,\pi\,dR$ and $dR = \dfrac{\pm\, 0.1}{24\,\pi} = \pm\, 0.0013.$

Therefore the error in the measured value of the diameter must not exceed 0.0026 in. if the error in the calculated value of the volume is not to exceed 0.1 cu. in.

PROBLEMS

1. Find an approximate formula for the area of a circular ring of radius r and width dr. What is the exact formula?

2. If A is the area of a square of side x, find dA. Draw a figure showing geometrically the square, dA, and ΔA.

3. Find an approximate formula for the volume of a thin cylindrical shell with open ends if the radius is r, the length h, and the thickness t.

4. The acceleration (a) due to gravity varies inversely as the square of the distance (s) from the center of the earth and is 32.2 at the surface of the earth (that is, $a = 32.2$ when s equals the radius of the earth, which is about 21,000,000 ft.). Find the formula for a in terms of s and the change in a produced by going up 21,000 ft. in a balloon.

5. A box is to be constructed in the form of a cube to hold 8 cu. ft. How accurately must the inner edge be made so that the volume will be correct to within 10 cu. in.?

6. The time of one vibration of a pendulum is given by the formula

$$t^2 = \frac{\pi^2 l}{g},$$

where t is measured in seconds, $g = 32.2$, and l, the length of the pendulum, is measured in feet. Find (a) the length of a pendulum vibrating once a second, (b) the change in t if the pendulum in (a) is lengthened 0.01 ft., (c) how much a clock with this error would lose or gain in a day.

7. If $y = x^{\frac{3}{2}}$ and the possible error in measuring x is 0.2 when $x = 16$, what is the possible error in the value of y? Use this result to obtain approximate values of $(16.2)^{\frac{3}{2}}$ and $(15.8)^{\frac{3}{2}}$.

8. Use differentials to find an approximate value of $\dfrac{1}{\sqrt{25.5}}$.

9. Show by means of differentials that

$$\frac{1}{x + dx} = \frac{1}{x} - \frac{dx}{x^2} \text{ (approximately).}$$

10. The reciprocal of 25 is 0.04. Find by differentials a good approximation to the reciprocal of 24.7.

11. What is the effect upon the volume of a cylinder if there is a possible error of 0.05 in. in measuring the radius and the altitude, which are found to be 8 in. and 12 in. respectively?

12. The brake horse power of an engine with m cylinders of diameter d in., according to the rating of the Association of Automobile Manufacturers, is given by the formula $P = \frac{md^2}{2.5}$. Approximate the increase in horse power of a four-cylinder engine obtained by increasing the diameter of each cylinder from $3\frac{1}{4}$ to $3\frac{3}{8}$ in.

13. The volume of a sphere is increasing at the rate of 16 cu. in./sec. Approximate the increment in the radius in the next $\frac{1}{2}$ sec. after the radius becomes 6 in.

88. Relative Error. In judging the accuracy of measured quantities or of others dependent upon them, the actual value of the possible error is usually of less importance than the *relative error.* The relative error is the ratio of the actual error to the quantity under consideration. For example, if a mile is measured with a possible error of 1 ft., the relative error is $\frac{1}{5280} = 0.00019$, and this measurement is relatively as accurate as that of a foot with a possible error of $0.00019 \times 12 = 0.0023$ in.

The relative error in a quantity is sometimes expressed as a percentage, and is known as the *percentage error.* In the example just considered, the percentage error was 0.019 per cent, — less than two hundredths of 1 per cent.

By definition the true value of the relative error in a value of y may be expressed as $\frac{\Delta y}{y}$; but unless the measurements are grossly inaccurate, Δy is nearly equal to dy, and the relative error is therefore closely approximated by the value of $\frac{dy}{y}$.

Example 1. The radius of a sphere is measured as 3 in., with a possible error of 0.02 in. Find (*a*) the greatest possible error in the calculated volume, (*b*) the greatest possible relative error in the calculated volume.

Solution. (*a*) We are given $r = 3$ and $dr = 0.02$. We are required to find dV. Differentiating $V = \frac{4\pi r^3}{3}$, we obtain

$$dV = 4\pi r^2 dr.$$

Substituting the values of r and dr, we obtain

$$dV = (4\pi)(9)(0.02) = 0.72\pi = 2.26 \text{ cu. in.}$$

(*b*) To obtain the relative error in V, we may proceed in two ways. The first is to calculate the value of V for $r = 3$, which is 36π cu. in. The relative error is then

$$\frac{dV}{V} = \frac{0.72\pi}{36\pi} = 0.02 = 2 \text{ per cent.}$$

It is usually better, however, to find $\frac{dV}{V}$ in terms of r and dr before substituting. By formula,

$$V = \frac{4\pi r^3}{3} \quad \text{and} \quad dV = 4\pi r^2 dr.$$

Dividing dV by V, we find

$$\frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4\pi r^3}{3}} = \frac{3\,dr}{r} = \frac{3(0.02)}{3} = 0.02 = 2 \text{ per cent.}$$

Example 2. Show that the relative error in the product of two numbers may be as great as the sum of the relative errors of the numbers.

Solution. Let the numbers be u and v and let y be their product; that is, $y = uv$. We are required to investigate $\frac{dy}{y}$.
Differentiating, we have

$$dy = u\,dv + v\,du.$$

The relative error in y is obtained by dividing by $y = uv$:

$$\frac{dy}{y} = \frac{u\,dv + v\,du}{uv} = \frac{dv}{v} + \frac{du}{u}.$$

But $\frac{dv}{v}$ and $\frac{du}{u}$ are the relative errors in v and u respectively. Hence we have proved that the relative error in the product is the sum of the relative errors of the factors if du and dv are both positive or both negative.

The result just obtained is of some importance in calculation. It shows that the product may be less accurate than either of the factors forming it. For example, if u is 2 per cent too large and v is 3 per cent too large $\left(\text{that is, if } \frac{du}{u} = 0.02 \text{ and } \frac{dv}{v} = 0.03\right)$, then y will be about 5 per cent too large. Only in the case where one number is too large and the other too small will the errors tend to compensate each other.

PROBLEMS

1. How exactly must the diameter of a circle be measured in order that the area shall be correct to within 1 per cent?

2. Show that the relative error in the volume of a sphere, due to an error in measuring the radius, is three times the relative error of the radius.

3. Show that if $y = \sqrt{x}$ the relative error in y due to an error in x is one half the relative error in x.

4. Show that the relative error in the nth power of a number is n times the relative error in the number.

5. Show that the relative error in the nth root of a number is $\frac{1}{n}$ times the relative error in the number.

6. When a cubical block of metal is heated, each edge increases $\frac{1}{10}$ per cent per degree increase in temperature. Show that the surface increases $\frac{2}{10}$ per cent per degree, and that the volume increases $\frac{3}{10}$ per cent per degree.

7. The value of g may be found by timing the vibration of a pendulum. Find the relative error in g due to a relative error of 1 per cent in measuring the time of vibration of a pendulum.

HINT. First solve the formula, given in Problem 6 (p. 183), for g in terms of t and l.

8. The value of g is found by timing the vibrations of a pendulum whose length was measured as 7.34 ft., with an uncertainty of 0.005 ft. The time of each vibration was 1.5 sec., which was assumed to be exact. Find the value of g, the greatest possible error in this value, and the greatest possible percentage error. (See hint in Problem 7.)

9. The boiling-point of water at altitude H(ft.) above sea level is given by $H = 517(212° - T) - (212° - T)^2$, T being the boiling temperature in degrees Fahrenheit. Find the uncertainty in the calculated value of H if the uncertainty in the measured value of T is 1° and T is measured as 200°.

89. Velocity along a Curve. When a point moves along a curve, as in Example 3 of § 82, with a constant or variable speed, it is desirable to find the rates at which x and y decrease or increase. For the position of the point at any instant is given by its coördinates x and y, and the measurement of lengths along a curve is not an easy process.

We must first distinguish between the terms *velocity* and *speed*. In mechanics the term *speed* is used to denote how fast an object is moving, while *velocity* denotes its speed and direction. Hence an object moving along a curve cannot have a constant velocity (since its direction is changing), but may have a constant speed. The speed of an object at a point is the magnitude of its velocity, while the inclination of the curve at that point gives the direction of the velocity.

Let us denote the distance measured along the curve from a fixed point A to the variable point $P(x, y)$ by s. If v denotes the speed of P, clearly v is the time rate at which s changes, or $v = \frac{ds}{dt}$. The rate at which x changes, or $\frac{dx}{dt}$, is denoted by v_x and is called the *x-component* of the velocity. Likewise $v_y = \frac{dy}{dt}$ is called the *y-component* of the velocity. We now proceed to find the equation connecting these quantities. Obviously v_x and v_y depend upon the direction as well as the magnitude of v.

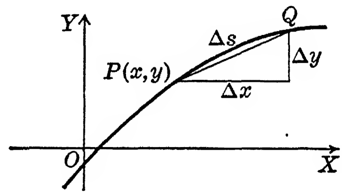


FIG. 87

Theorem. *If v denotes the velocity of a point along a curve, and if v_x and v_y are its x-component and y-component, then $v^2 = v_x^2 + v_y^2$.*

Proof. In the above figure let the moving point $P(x, y)$ describe the distance s along the curve from the fixed point A in time t. According to the above definitions we must prove that

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2. \tag{1}$$

Let Q be the position of P at time $t + \Delta t$; its coördinates may be denoted by $(x + \Delta x, y + \Delta y)$. The arc PQ is the corresponding increment of s and may be denoted by Δs. Let PQ always denote the chord.

We cannot express Δs directly in terms of Δx and Δy, but it is obvious that

$$(PQ)^2 = (\Delta x)^2 + (\Delta y)^2.$$

Dividing both sides by $(\Delta t)^2$, we have

$$\left(\frac{PQ}{\Delta t}\right)^2 = \left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2. \tag{2}$$

Now $\frac{\Delta s}{\Delta t} = \frac{\Delta s}{PQ}\frac{PQ}{\Delta t}$. We shall assume as axiomatic that as Δt, and in consequence PQ, approaches 0, $\lim \frac{\Delta s}{PQ} = 1$. Hence we have

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s}{PQ} \lim_{\Delta t \to 0} \frac{PQ}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{PQ}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}. \qquad \text{By (2)}$$

But, by the definition of *derivative*, $\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$, $\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$, and $\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}$. Hence this last equation gives

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

Squaring both sides, we have equation (1), which was to be proved.

Direction of Motion. We should also note the additional fact that if m denotes the slope of the curve at any point,

$$m = \frac{v_y}{v_x}.$$

For

$$\frac{v_y}{v_x} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dt}\frac{dt}{dx} \qquad \text{by § 59}$$

$$= \frac{dy}{dx} \qquad \text{by § 56}$$

$$= m. \qquad \text{By § 44}$$

The applications of the relations above are far-reaching. We can at once find the component velocities if the velocity along the curve is known, or we can determine the

velocity along the curve if we know one of the component velocities, provided, of course, that the path of motion is also known.

EXAMPLE 1. A point moves along the curve whose equation is $4y = 8x - x^2$ (the unit of distance being 1 ft.), with a velocity such that its x-component is constantly equal to 2 ft./min. Find its velocity along the curve at the point (6, 3).

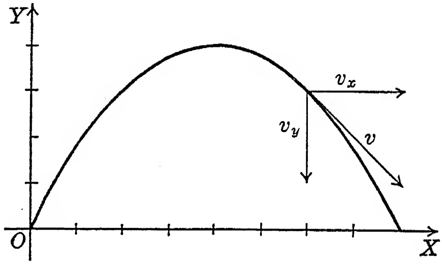


FIG. 88

Solution. Differentiating with respect to t, we have

$$4\frac{dy}{dt} = 8\frac{dx}{dt} - 2x\frac{dx}{dt}.$$

Substituting the given values of x and $v_x = \frac{dx}{dt}$, we obtain

$$4\frac{dy}{dt} = 8 \times 2 - 2 \times 6 \times 2 = -8;$$

that is,

$$v_y = \frac{dy}{dt} = -2 \text{ ft./min.}$$

By the above theorem,

$$v^2 = v_x^2 + v_y^2;$$

hence

$$v = \sqrt{4+4} = 2.828 \text{ ft./min.}$$

We note also that the slope of the curve at this point is -1, since $\frac{v_y}{v_x} = -1$.

EXAMPLE 2. A point moves in a counterclockwise direction along the circle $x^2 + y^2 = 100$ (the unit of distance being 1 in.), with a constant speed of 6 in./min. Find the components of its velocity at the point (6, 8).

Solution. Differentiating the given equation with respect to t, we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

Substituting the given values of x and y and solving for $\frac{dy}{dt}$, we obtain

$$\frac{dy}{dt} = -\frac{3}{4}\frac{dx}{dt},$$

or
$$v_y = -\frac{3}{4}v_x.$$

But we are given $v = 6$. Hence

$$v_x^2 + v_y^2 = 36.$$

Substituting for v_y,

$$v_x^2 + \tfrac{9}{16}v_x^2 = 36,$$

or
$$v_x = \sqrt{\frac{16 \times 36}{25}} = \pm 4.8.$$

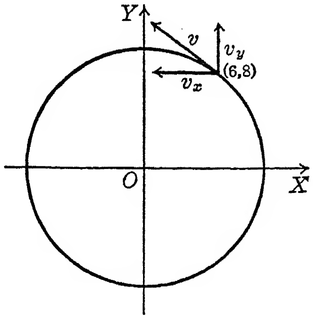


FIG. 89

Since the motion is counterclockwise, $v_x = -4.8$. Substituting this above, we find that $v_y = +3.6$ in./min.

90. Parametric Equations. Since v and its components are derivatives with respect to the time, it is convenient to have the coördinates of the point tracing the curve expressed as functions of t. When each coördinate is expressed as a function of any third variable t, the equations are called the *parametric equations* of the curve, and t is called the *parameter*. To plot such a locus, simply compute the pairs of values of x and y corresponding to various values of t and plot the points as usual. To reduce the equation to the usual form eliminate the parameter between the equations.

EXAMPLE. A point moves according to the laws $x = t^2 + 2$, $y = \frac{t^3}{3}$.

(*a*) Plot the locus.

(*b*) Find the position and velocity of the point and its direction of motion after 3 sec.

(*c*) Eliminate the parameter.

Solution. The adjacent table of values is obtained by substituting in the equations. The dotted portion of the curve corresponds to negative values of t, which may or may not have a meaning.

t	x	y
0	2	0
± 1	3	$\pm \frac{1}{3}$
± 2	6	$\pm \frac{8}{3}$
± 3	11	± 9
± 4	18	$\pm \frac{64}{3}$

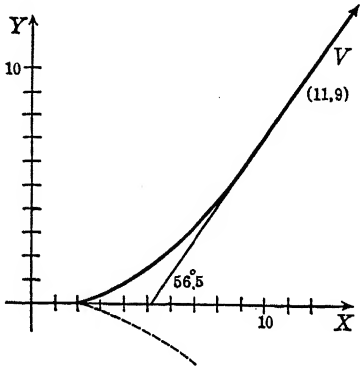


FIG. 90

When $t = 3$ the table shows that the position of the point is (11, 9). To find its velocity, differentiate each equation with respect to t, and substitute $t = 3$. We obtain

$$v_x = 2t = 6$$

and $$v_y = t^2 = 9.$$

Hence $$v = \sqrt{36 + 81} = \sqrt{117} = 10.8.$$

To find its direction of motion, use the fact that $m = \dfrac{v_y}{v_x}$. This gives $m = \frac{3}{2}$; or the point is moving upward in a direction inclined at 56.31° (56° 19′) with the positive half of the x-axis.

To eliminate the parameter, solve the first equation for t in terms of x, getting $t = \sqrt{x - 2}$, and substitute this in the second equation:

$$y = \frac{(x-2)^{\frac{3}{2}}}{3}.$$

The curve could, of course, be plotted from this last equation, but with more difficulty.

PROBLEMS

1. Plot the following parametric equations:

a. $x = 3t,\ y = \dfrac{1}{6t}.$

b. $x = t^2,\ y = 2t.$

c. $x = t + 1,\ y = \dfrac{t^3}{2}.$

d. $x = 2t,\ y = 3t - 5.$

e. $x = t^2,\ y = t^3.$

f. $x = 1 - t^2,\ y = 2t.$

g. $x = t^2 + 1,\ y = t^3.$

h. $x = 16t,\ y = 16t - t^2.$

i. $x = t + \dfrac{1}{t},\ y = t - \dfrac{1}{t}.$

j. $x = (t-2)^3,\ y = t.$

2. Eliminate the parameter in each pair of equations in Problem 1.

3. In each curve in Problem 1 find the speed and direction of motion when $t = 2$.

4. A boat which can steam 16 mi./hr. is heading due north across an ocean current setting east at 3 mi./hr. In what direction is the boat really going and how fast?

5. A point moves upward from the point (6, 0) along the straight line whose equation is $4x - 3y = 24$, at a speed of 16 ft./sec. Find v_x and v_y, and express x and y in terms of t as a parameter.

6. A point describes each of the following curves. Find its speed and direction of motion in each case, and illustrate by a figure:

a. $9x^2 + 4y^2 = 100,\ v_x = -1,\ y = 4.$

b. $8y = x^3,\ v_x = 3,\ x = -2.$

c. $4y^3 = x^2,\ v_y = -2,\ y = 4.$

d. $y(4 + x^2) = 8,\ v_x = 2,\ x = 0$, and $x = 2$.

7. A moving point describes the curve whose equation is $y = x^3 - 3x^2 - 9x - 4$ with a constant speed of 16 ft./sec. Find the points where $v_y = 0$. What is v_x at these points? Sketch the curve.

8. A moving point describes the curve $y^2 = 8x$ with a constant speed of 4 ft./sec. Find v_x, v_y, and the direction of motion at the point $(2, -4)$.

9. A point moves counterclockwise about the circle whose equation is $x^2 + y^2 = 36$. Find the two points at which $v_x = v_y$.

10. A moving point describes the curve whose equation is $y = 16 + 4x - x^2$ with a constant speed of $v = 16$ ft./sec. Find the point where $v_x = 8$ ft./sec.

11. The slope of a curve at a certain point is $-\frac{4}{3}$, and the speed of the point describing it is 24 ft./sec. at that point. Find v_x and v_y.

12. A projectile is hurled at an angle with the horizontal whose tangent is $\frac{3}{4}$, with an initial speed of 200 ft./sec. What are the values of v_x and v_y initially? Which of these remains constant if air resistance is disregarded? In the light of the note at the end of § 81, write a formula giving v_y at any time.

CHAPTER VII

THE CIRCLE, PARABOLA, ELLIPSE, AND HYPERBOLA

THE CIRCLE

91. The Standard Equation of the Circle. In the first chapter it was shown that the equation of every straight line is of the first degree and, conversely, that the locus of every equation of the first degree is a straight line. In a similar way we shall now find the form of the equation of every circle.

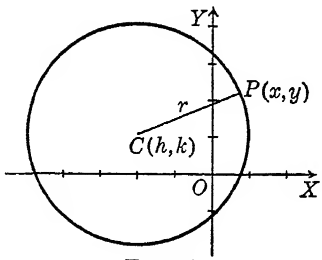


FIG. 91

Let $P(x, y)$ be any point on a circle whose radius is r and whose center is $C(h, k)$. Since, by the definition of a circle, the radius is constant, we have at once, for all positions of P,

$$CP = r;$$

whence $$\sqrt{(x-h)^2+(y-k)^2}=r,$$

or

$$\text{(I)} \qquad (x-h)^2+(y-k)^2=r^2.$$

Equation I is the *standard equation* of a circle, and may be used to write the equation of any circle when its center and radius are known. In the above figure the center is $(-2, 1)$ and the radius is 3. Hence the equation is

$$(x+2)^2+(y-1)^2=9.$$

The position and size of the circle depend upon the three arbitrary constants h, k, and r. Since h and k are merely the coördinates of the center, they may be either positive or negative, while r is necessarily positive.

If the center is at the origin, $h=0$ and $k=0$, and form I reduces to

$$x^2+y^2=r^2. \tag{I a}$$

This is usually employed when it is desired to study some property of the circle which does not depend upon its position.

92. The General Form of the Equation of the Circle. If we expand the parentheses in equation I and collect the terms, we get

$$x^2+y^2-2\,hx-2\,ky+(h^2+k^2-r^2)=0.$$

Since $-2\,h$, $-2\,k$, and $h^2+k^2-r^2$ are constants, this may be written in the form

$$x^2+y^2+Dx+Ey+F=0, \tag{II}$$

which may be called the *general form.* If the equation of a circle is given in this form, it can be reduced back to form I by merely completing the squares in x and y.

The most general equation of the second degree in x and y is

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0.$$

This equation can be reduced to the general form of the equation of a circle if, and only if, $B=0$ and $C=A$. When $B=0$ and $C=A\neq 1$, the equation can be reduced to form II by merely dividing through by the common value of A and C. Thus we have proved the following theorem:

Theorem. *The locus of an equation of the second degree is a circle if, and only if, the coefficients of x^2 and y^2 are equal and there is no xy term.* (See the apparent exception in Example 2 below.)

93. To find the Center and Radius of a Circle. As stated in the preceding section, if the equation of a circle is given in the general form it can be reduced to the standard form by completing the squares in x and y. The center and the radius can then be identified at once.

EXAMPLE 1. Find the center and the radius of the circle whose equation is

$$2x^2 + 2y^2 + 8x - 6y + 7 = 0.$$

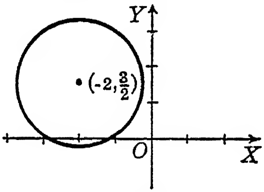


FIG. 92

Solution. Dividing through by 2, we have the general form

$$x^2 + y^2 + 4x - 3y + \tfrac{7}{2} = 0.$$

To find the center and the radius, reduce this to form I by transposing $\frac{7}{2}$ and completing the squares.

$$x^2 + 4x + 4 + y^2 - 3y + \tfrac{9}{4} = -\tfrac{7}{2} + 4 + \tfrac{9}{4} = \tfrac{11}{4},$$

$$(x+2)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{11}{4} = \left(\frac{\sqrt{11}}{2}\right)^2.$$

Comparing this with the standard form, we see that $h = -2$, $k = \dfrac{3}{2}$, and $r = \dfrac{\sqrt{11}}{2} = 1.66$, approximately. Hence the center is $(-2, \frac{3}{2})$, and the radius is about 1.66.

The algebraic work is so simple that a check is scarcely necessary; but if one is desired, the simplest is to find one or more points from the given equation and compare them with the figure. Here the x-intercepts are

$$-2 \pm \frac{\sqrt{2}}{2} = -2 \pm 0.71 = -2.71 \text{ or } -1.29.$$

EXAMPLE 2. What is the locus of $x^2 + y^2 - 8x + 4y + 21 = 0$?

Solution. According to the preceding theorem it is a circle. Completing the squares as before, we have

$$(x-4)^2 + (y+2)^2 = -21 + 16 + 4 = -1.$$

But here $r^2 = -1$, making r imaginary. An inspection of the equation shows that the locus is imaginary, as $(x-4)^2$ and $(y+2)^2$ are squares and therefore never less than zero; hence their sum cannot equal -1.

To avoid having exceptions we say that the locus is an *imaginary circle.* When $r = 0$, making the locus a point, we call it a *point-circle.* Such forms are called *degenerate forms.*

94. The Slope of the Tangent. As in the case of other curves, we find the slope of the tangent to a circle by differentiating its equation.

It is also easy to show that the tangent is perpendicular to the radius drawn to the point of contact. For by differentiating the standard equation

$$(x-h)^2 + (y-k)^2 = r^2,$$

we have
$$2(x-h) + 2(y-k)\frac{dy}{dx} = 0;$$

whence
$$m = \frac{dy}{dx} = -\frac{x-h}{y-k}.$$

But the slope m' of the radius joining the center $C(h, k)$ to the point $P(x, y)$ on the circumference is

$$m' = \frac{y-k}{x-h}.$$

Hence $mm' = -1$, which is the criterion for perpendicularity.*

In problems involving the tangent to a circle the slope may be found either by differentiating or by using the negative reciprocal of the slope of the radius. The former method is more convenient when the center is not given.

*We have now reconciled the definition of tangent given in Chapter III with that given in plane geometry. For the tangent as defined in plane geometry is there shown to be perpendicular to the radius to the point of contact; and as there can be but one perpendicular to a line at a given point, both definitions give the same line.

EXAMPLE. Find the equation of the tangent to the circle $x^2 + y^2 - 4x - 2y = 20$ at the point (5, 5).

Solution. Differentiating the equation, we find

$$m = \frac{dy}{dx} = \frac{2 - x}{y - 1}.$$

Substitution of the coördinates (5, 5) gives $m = -\frac{3}{4}$. Hence the equation of the tangent is

$$y - 5 = -\tfrac{3}{4}(x - 5), \quad \text{or} \quad 3x + 4y = 35.$$

PROBLEMS

1. In each of the following cases write the equation of the circle and reduce it to the general form:

a. Center (6, 4), radius 6.
b. Center (4, 3), radius 5.
c. Center (5, −12), radius 13.
d. Center $(a, 0)$, radius a.
e. Center $(0, a)$, radius a.
f. Center (a, a), radius $a\sqrt{2}$.

2. Find the center and the radius and draw each of the following circles:

a. $(x + 2)^2 + (y - 3)^2 = 25$.
b. $(x - 1)^2 + (y + 4)^2 = 0$.
c. $x^2 + y^2 - 8x + 6y + 24 = 0$.
d. $x^2 + y^2 - 13x = 0$.
e. $2x^2 + 2y^2 + 15y = 0$.
f. $4x^2 + 4y^2 - 4x - 4y + 1 = 0$.
g. $4x^2 + 4y^2 - 12x - 81 = 0$.
h. $3x^2 + 3y^2 + 36x - 14y = 0$.
i. $x^2 + y^2 - 20x + 40y + 379 = 0$.
j. $6x^2 + 6y^2 = 25y$.

3. Determine by inspection which of the circles in Problem 2

a. Have their centers on the x-axis.
b. Have their centers on the y-axis.
c. Pass through the origin.

4. Find the equations of the circles satisfying the following conditions and draw the figure in each case:

a. Center at $(1, 2)$ and passing through $(-2, 3)$.

b. Having the line joining $(2, -4)$ and $(4, 6)$ for a diameter.

c. Center at $(-3, 5)$ and tangent to the x-axis.

d. Center at $(4, -2)$ and tangent to the y-axis.

e. Center on the line $y=7$, radius 2, and tangent to the y-axis.

f. Center on the line $x+2=0$, radius 3, and tangent to the x-axis.

g. Radius 5 and tangent to both axes.

h. Center on the line $x=5$, radius 13, and passing through the origin.

i. Center at the origin and tangent to the line $x+y=6$.

j. Center at $(1, 2)$ and tangent to the line $x-y-4=0$.

5. Draw the system of circles defined by $(x+4)^2+(y-3)^2=k$ for the following values of k: $k = 25, 16, 9, 4, 1, 0, -1$.

6. *a.* The point $(\frac{7}{2}, \frac{1}{2})$ bisects a chord of the circle $x^2+y^2=25$. Find the equation of the chord, and its length.

b. Solve the same problem for the point $(4, 7)$ and the circle $x^2+y^2-4x-6y=12$.

7. Find the locus of the vertex of a right triangle which has the ends of its hypotenuse at $(0, -3)$ and $(6, 5)$.

8. Find the locus of a point the sum of the squares of whose distances from the points $(\pm c, 0)$ is k. For what values of k does the locus fail to exist?

9. The ends of the base of a triangle are the points $(\pm c, 0)$. Find the equation of the locus of the vertex if the median to one of the sides has a constant length k.

10. Find the equation of the locus of a point whose distance from the origin is always twice its distance from $(6, 0)$. Draw the locus.

11. Find the equation of the locus of a point whose distance from $(-2, 2)$ is always three times its distance from $(2, -2)$. Draw the locus.

12. Show that the locus of a point whose distance from $(c, 0)$ is k times its distance from $(-c, 0)$ is a circle. Find its center and radius.

13. Find the equations of the tangent and of the normal to each of the following circles at the point indicated:

a. $(x+2)^2+(y-3)^2=25$, $(2, 0)$.
b. $x^2+y^2-8x+6y+24=0$, $x=4$.
c. $x^2+y^2-13x=0$, $x=4$.
d. $4x^2+4y^2-4x-4y+1=0$, $(0.8, 0.9)$.
e. $4x^2+4y^2-12x-81=0$, $x=6$.
f. $3x^2+3y^2+36x-14y=0$, $(0, 0)$.
g. $6x^2+6y^2=25y$, $(-\frac{5}{3}, \frac{10}{3})$.

14. Find the angles of intersection of each of the following circles and straight lines:

a. $(x+2)^2+(y-3)^2=25$, $7x-y-8=0$.
b. $x^2+y^2-8x+6y+24=0$, $3x+4y=0$.
c. $x^2+y^2-13x=0$, $y=x$.
d. $4x^2+4y^2-4x-4y+1=0$, $2x-14y+11=0$.
e. $4x^2+4y^2-12x-81=0$, $4x-2y-21=0$.
f. $3x^2+3y^2+36x-14y=0$, $x+y=0$.
g. $6x^2+6y^2=25y$, $y=3x$.

15. Find the angles of intersection of each of the following pairs of circles:

a. $x^2+y^2+4y=5$, $x^2+y^2-4x=1$.
b. $x^2+y^2-8y-1=0$, $x^2+y^2-10x-2y+9=0$.

16. Find the angle of intersection of two circles of equal radii, the center of each of which is on the circumference of the other.

17. Two vertices of a regular hexagon of side a are $(-a, 0)$ and $\left(\frac{a}{2}, \frac{a\sqrt{3}}{2}\right)$. Circles of radius a are drawn with each of these points as centers. Find their angles of intersection.

18. A rectangle is circumscribed about a semicircle. Find the angles of intersection of its diagonals with the semicircle.

95. Note on the Analytic Method of Solving Problems. In the problems of this chapter the methods developed previously should be freely used. These have been mainly analytical in character, but in solving the easier problems there is danger of forgetting the fundamental principle involved; namely, that *the coördinates of every point on the locus must satisfy the equation.* In certain problems a keen realization of this principle is essential. This will be illustrated in the two following examples:

EXAMPLE 1. Find the dimensions of the largest cylinder that can be cut from a spherical segment which is cut off from a sphere of radius 5 in. by a plane 2 in. distant from the center of the sphere.

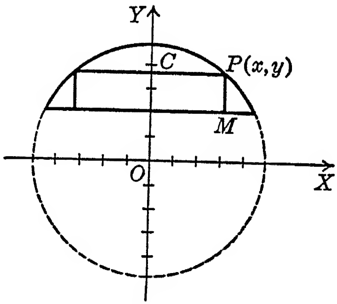


FIG. 93

Solution. The adjoining figure represents a section made by a plane passing through the axis of the cylinder, which we here take as the y-axis. The section of the cylinder is a rectangle of which one vertex $P(x, y)$ lies on the circle. We easily see that the radius of the cylinder is

$$CP = x$$

and that the altitude is

$$MP = y - 2.$$

Hence the volume which is to be a maximum is given by the formula

$$V = \pi x^2 (y - 2).$$

Since P lies on the circle, its coördinates must satisfy the equation of the circle, which is

$$x^2 + y^2 = 25.$$

Using this relation, we find that

$$V = \pi(25 - y^2)(y - 2)$$
$$= \pi(25\,y - 50 - y^3 + 2\,y^2),$$

which satisfies the requirement of expressing V in terms of one variable.

Differentiating, we have

$$\frac{dV}{dy} = \pi(25 - 3\,y^2 + 4\,y).$$

Setting this equal to zero and solving the resulting equation, we get

$$y = \frac{4 \pm \sqrt{316}}{6}.$$

The negative result obviously has no meaning; the positive sign gives $y = 3.63$, approximately. Hence the altitude of the cylinder is $y - 2 = 1.63$, and the radius is $x = \sqrt{25 - y^2} = 3.44$.

EXAMPLE 2. Find the equation of the tangent to the circle $x^2 + y^2 - 4\,x - 2\,y = 20$ which is parallel to the line $4\,x + 3\,y = 18$.

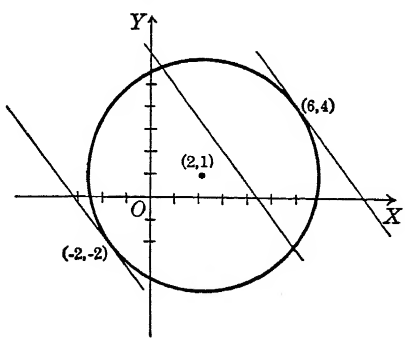


FIG. 94

Solution. Differentiating the equation, we find, for the slope of the tangent at any point (x, y) on the circle,

$$m = \frac{dy}{dx} = \frac{2 - x}{y - 1}.$$

But the slope of the parallel line, and therefore that of the required tangent, is $-\frac{4}{3}$. Hence the coördinates of the point of tangency satisfy the equation

$$\frac{2 - x}{y - 1} = -\frac{4}{3}.$$

But they must also satisfy the equation of the circle. They may therefore be obtained by solving the two equations simultaneously. The algebraic work follows:

$$6 - 3x = -4y + 4,$$

$$x = \frac{4y+2}{3}.$$

Substituting this in the given equation, we obtain

$$\left(\frac{4y+2}{3}\right)^2 + y^2 - 4\left(\frac{4y+2}{3}\right) - 2y = 20,$$

which reduces to $\qquad y^2 - 2y - 8 = 0.$

Thus $\qquad y = 4 \quad \text{or} \quad -2,$

and $\qquad x = 6 \quad \text{or} \quad -2.$

Thus there are two tangents, with points of contact (6, 4) and (− 2, − 2), and their equations are

$$y - 4 = -\tfrac{4}{3}(x - 6) \quad \text{and} \quad y + 2 = -\tfrac{4}{3}(x + 2),$$

or $$4x + 3y = 36 \quad \text{and} \quad 4x + 3y = -14.$$

96. The Circle Determined by Any Three Conditions. In Chapter I we noticed that a straight line is determined by two points upon it and that its standard equations each involve two independent arbitrary constants. Since the equation of the circle contains three arbitrary constants, we should be able to find its equation if we are given three points upon it or any other three geometrical conditions determining it.

The simplest case is where three points are given; but the method of procedure is the same in all cases. Express the geometrical conditions in the form of three equations having as unknowns h, k, and r (or D, E, and F), and solve these equations simultaneously for the unknowns. Then write down the equation of the circle.

EXAMPLE 1. Find the equation of the circle passing through $(7, 1)$, $(6, 8)$, and $(-1, 7)$.

Solution. Each of these pairs of coördinates must satisfy the equation of the circle. Therefore, using the standard form of the equation,

$$(7-h)^2+(1-k)^2=r^2,$$
$$(6-h)^2+(8-k)^2=r^2,$$
$$(-1-h)^2+(7-k)^2=r^2.$$

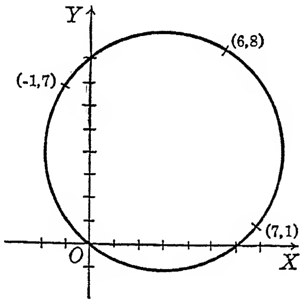


FIG. 95

These equations are easy to solve simultaneously; for if we expand them and subtract the first successively from the second and third, we eliminate all terms of the second degree, having left

$$2h-14k=-50,$$
$$16h-12k=0.$$

These give $k=4$, $h=3$, and $r=5$. Therefore the desired equation is

$$(x-3)^2+(y-4)^2=5^2,$$

which may be reduced to

$$x^2+y^2-6x-8y=0.$$

If we use the general form of the circle equation, we obtain

$$49+1+7D+E+F=0,$$
$$36+64+6D+8E+F=0,$$
$$1+49-D+7E+F=0.$$

Solving, $\quad D=-6,\quad E=-8,\quad$ and $\quad F=0.$

In problems of this kind the circle should be drawn with the correct radius and center. As a check upon the work the coördinates of the given points should be substituted in the final equation. When three points are given, this check is absolute.

EXAMPLE 2. Find the equation of the circle which is tangent to the line $3x - 4y = 2$ at the point (2, 1) and which passes through the origin.

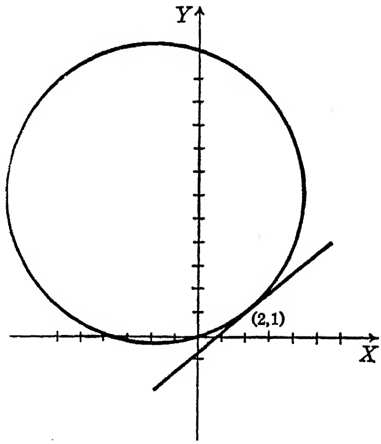


FIG. 96

Solution. There are given two points on the circle, (2, 1) and (0, 0). Substituting these coördinates in the general form of the equation of a circle, we have

$$5 + 2D + E + F = 0, \quad (1)$$

$$F = 0. \quad (2)$$

We need a third equation. This is provided by the fact that the given line is tangent to the circle at (2, 1), which makes the slope of the tangent at this point equal to $\frac{3}{4}$.

Differentiating the equation

$$x^2 + y^2 + Dx + Ey + F = 0,$$

we obtain
$$2x + 2y\frac{dy}{dx} + D + E\frac{dy}{dx} = 0,$$

or
$$\frac{dy}{dx} = -\frac{2x + D}{2y + E}.$$

At the point (2, 1) this is equal to $-\dfrac{4 + D}{2 + E}$.

Hence the third equation is

$$-\frac{4 + D}{2 + E} = \frac{3}{4},$$

or
$$4D + 3E = -22. \quad (3)$$

Solving equations (1), (2), and (3) simultaneously, we obtain

$$D = \tfrac{7}{2}, \quad E = -12, \quad F = 0.$$

Hence the equation of the circle is

$$x^2 + y^2 + \tfrac{7}{2}x - 12y = 0.$$

When reduced to the standard form, this equation becomes

$$(x + \tfrac{7}{4})^2 + (y - 6)^2 = \tfrac{625}{16},$$

which shows that the center is $(-\tfrac{7}{4}, 6)$ and the radius is $\tfrac{25}{4}$.

PROBLEMS

1. Find the equation of the circle determined by the following points:

a. $(4, 2)$, $(2, 4)$, $(-4, 2)$.

b. $(1, 3)$, $(5, 1)$, $(3, -3)$.

c. $(8, 0)$, $(0, 12)$, $(7, 5)$.

d. $(1, -8)$, $(9, -4)$, $(10, -5)$.

e. $(2, 1)$, $(2, 9)$, $(10, 5)$.

f. $(6, 3)$, $(0, 6)$, $(-6, -6)$.

2. Find the equation of the circle which

a. Has the center $(4, 1)$ and passes through $(-2, 5)$.

b. Has the line joining $(-2, 5)$ and $(7, 4)$ as a diameter.

c. Passes through $(0, 4)$ and $(6, 8)$ and has its center on the x-axis.

d. Passes through $(2, 5)$ and $(10, 1)$ and has its center on the line $x - y - 4 = 0$.

e. Passes through $(16, 12)$ and $(2, -2)$ and is tangent to the y-axis.

f. Passes through the point $(4, 4)$ and is tangent to the line $x - y - 4 = 0$ at $(4, 0)$.

g. Passes through $(18, -25)$ and is tangent to both axes.

h. Has its center at $(-2, 4)$ and is tangent to the line $x - y - 6 = 0$.

3. Find the coördinates of the points of contact of the tangents drawn from the given point to the given circle:

a. $x^2 + y^2 = 13$, $(5, -1)$.

b. $x^2 + y^2 - 10x = 0$, $(0, 10)$.

4. Tangents are drawn from $(10, -10)$ to the circle $x^2 + y^2 = 20$. Find the perimeter of the area bounded by the tangent lines and the circumference.

5. Find the dimensions of the largest rectangle that can be cut from a semicircle whose radius is a.

6. Find the length of the upper base of the largest trapezoid that can be cut from a semicircle whose radius is a.

7. Show that the largest isosceles triangle that can be inscribed in a circle is equilateral.

8. Find the dimensions of the largest rectangle that can be cut from the segment bounded by the circle whose equation is $x^2 + y^2 = 25$ and the line whose equation is $x = 2$.

9. Show that the equation of the tangent to the circle $x^2 + y^2 = r^2$ at any point (x_1, y_1) is $x_1x + y_1y = r^2$.

10. In Problem 9 show that the equation of the normal is $x_1y - y_1x = 0$.

11. A square is inscribed in a circle, and another circle is drawn with a vertex of the square as a center and a side of the square as a radius. Find the angle at which it meets the first circle.

12. The equation of a certain circle is $x^2 + y^2 + 6x + 4y = 87$. Find the equations of circles tangent to this circle and satisfying the following conditions:

a. Center at $(2, -2)$.

b. Point of tangency $(-13, -2)$, radius 6.

THE PARABOLA

97. Definition. The locus of a point equidistant from a given fixed point and a given fixed line is called a *parabola*.

The fixed point is called the *focus*; and the given fixed line is called the *directrix*. The distance between the directrix and the focus is denoted by p. The line perpendicular to the directrix and passing through the focus is called the *axis* of the parabola.

By definition of the parabola, the point on the axis halfway between the focus and the directrix lies upon the locus; it is called the *vertex*.

98. Construction of the Parabola. When the focus and directrix are given, the parabola may be constructed with ruler and compasses. Let DD' be the directrix and F the focus. The line AFB, perpendicular to DD', is the axis, and V, midway between A and F, is the vertex. Through any point C on the axis draw a line CC' parallel to the directrix. With AC as radius and F as center describe a circular arc cutting CC' in P and P'. The points P and P' are equidistant from F and from DD' and hence lie on the parabola. By choosing different positions of C on the axis it is a simple matter to get as many points as desired on the curve. Naturally the point C must be taken on the same side of the vertex as the focus, for it is obvious that no point on the parabola can be nearer the directrix than the vertex.

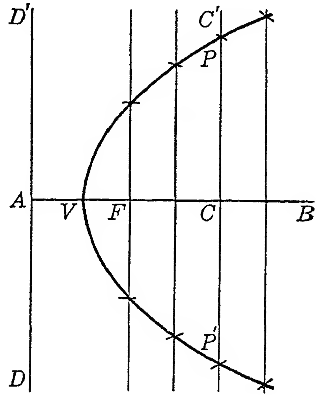


FIG. 97

The preceding construction is facilitated by the use of squared paper on which the lines CC' are drawn at regular intervals.

99. Equations of the Parabola. Any line may be taken as the directrix, and any point not upon the directrix may be taken as the focus. In general, however, the equations thus obtained are very complicated. We shall therefore direct our attention for the present to the equations obtained by taking the vertex at the origin and the axis of the parabola as the x-axis or y-axis.

Let us first take the axis of the parabola as the x-axis. The focus F will then lie upon the x-axis, and the directrix

AB will be parallel to the y-axis. Since the vertex bisects the segment of the axis between the focus and the directrix, the coördinates of F are $\left(\frac{p}{2}, 0\right)$, and the equation of AB is $x = -\frac{p}{2}$.

Now let $P(x, y)$ be any point on the locus. Join F and P and draw PM perpendicular to AB. By definition of the parabola,

$$FP = MP.$$

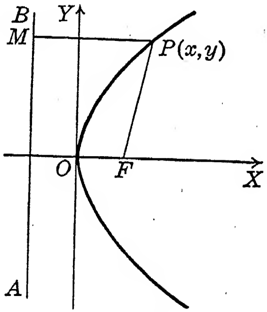


FIG. 98

By inspection and by use of the distance formula this becomes at once

$$\sqrt{\left(x - \frac{p}{2}\right)^2 + y^2} = x + \frac{p}{2}.$$

Squaring and simplifying, we have the standard equation

(III) $$y^2 = 2\,px.$$

If the focus is taken at the point $\left(0, \frac{p}{2}\right)$, and the directrix as the line $y = -\frac{p}{2}$, we obtain in the same way the form

(III a) $$x^2 = 2\,py.$$

Taking the focus at the left or below the directrix yields two other standard forms, as follows:

Focus $\left(-\frac{p}{2}, 0\right)$, directrix $x = \frac{p}{2}$:

(III b) $$y^2 = -2\,px.$$

Focus $\left(0, -\frac{p}{2}\right)$, directrix $y = \frac{p}{2}$:

(III c) $$x^2 = -2\,py.$$

EXERCISE. Derive equations III a, III b, and III c.

100. Discussion of the Equations. We shall discuss only one of the standard equations; the others are treated in a similar manner. Let us take

$$y^2 = 2\,px.$$

The form of the equation shows that the intercepts are both 0. From this we deduce that the parabola crosses its axis at the vertex alone.

Since the only term involving y is y^2, there is symmetry with respect to the x-axis.

Solving the equation for y in terms of x, $y = \pm\sqrt{2\,px}$. Hence x can never be negative; that is, the nearest point of the parabola to its directrix is the vertex. As x increases, y also increases. Hence the parabola recedes indefinitely from both the axis and the directrix.

The discussion makes it easy to distinguish between the various standard forms. For a form involving y^2 the axis of symmetry is the x-axis, and the focus is to the left or right of the vertex according to the sign before $2\,px$. Similar remarks apply to the forms containing x^2.

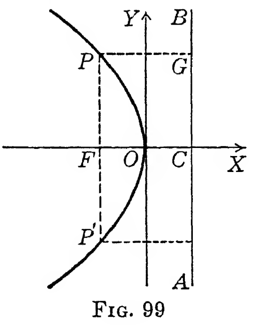


FIG. 99

101. The Latus Rectum. The chord through the focus of a parabola perpendicular to its axis is called the *latus rectum*. In the figure, F is the focus, ACB the directrix, and $P'P$ the latus rectum. Draw PG perpendicular to AB.

By definition of the parabola, the distance of P from the focus is the same as that from the directrix. Hence

$$FP = PG = FC = p,$$

and the length of the whole latus rectum is $2\,p$.

102. Drawing the Parabola. When the equation of a parabola is given, the curve may be plotted by using as many points as necessary for the degree of accuracy required. If it is desired merely to sketch the curve to show its position, size, and general shape, the method of the following example is usually sufficient:

EXAMPLE. Sketch the parabola $x^2 = -12y$.

Solution. This is of the form $x^2 = -2py$; hence $2p = 12$, or $p = 6$. Since the term in x is of the second degree and the sign before $2p$ is minus, the axis of the parabola is the y-axis and the curve lies wholly below the x-axis. Therefore the focus is $\left(0, -\frac{p}{2}\right)$, or $(0, -3)$; and the equation of the directrix is $y = 3$. Measuring off $p = 6$ to the right and the left of the focus gives the ends of the latus rectum. These, together with the vertex O, make three points on the curve, which suffice for a sketch.

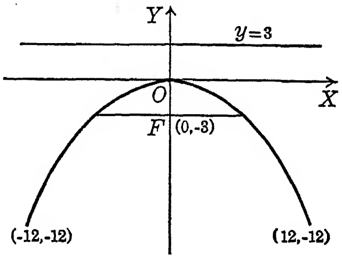


FIG. 100

The sketch may be improved with little trouble by also finding the ends of the chord $y = -2p$. These are $(\pm 2p, -2p)$.

PROBLEMS

1. With coördinate paper and compasses construct a parabola having (a) its focus 10 units from its directrix, (b) its latus rectum 40 units long.

2. Find the coördinates of the focus, the equation of the directrix, and the length of the latus rectum of each of the following parabolas. Sketch the curve.

a. $y^2 = 16x$. *c.* $2x^2 - 9y = 0$. *e.* $y^2 = 6ax$.
b. $x^2 + 8y = 0$. *d.* $3y = 4x^2$. *f.* $15x + 2y^2 = 0$.

3. Write the equations of the parabolas satisfying the given conditions and draw the figure in each case:

a. Directrix $y = 4$, focus $(0, -4)$.
b. Directrix $x = -8$, focus $(8, 0)$.
c. Directrix $y = -2$, vertex $(0, 0)$.
d. Latus rectum $= 8$, vertex $(0, 0)$.
e. Vertex $(0, 0)$, focus on x-axis, and passing through $(-2, 6)$.
f. Vertex $(0, 0)$, and passing through $(4, 6)$.

4. One end of a chord through the focus of a parabola which has the x-axis as the principal axis and the origin as the vertex is the point $(4, -4)$. Find the coördinates of the other end.

5. Find the equation of the circle passing through the vertex and the ends of the latus rectum of the parabola $y^2 = 8x$.

6. In Problem 5 find the angle between the circle and the parabola at each point of intersection.

7. Find the slope of the parabola $y^2 = 2px$ in terms of x and show that as x approaches infinity the slope approaches 0.

8. Show that the parabola $y^2 = 2px$ has no maximum or minimum point.

9. Find the equations of the tangent and the normal to each of the following parabolas at the point indicated:

a. $2y^2 = 3x$, $(6, 3)$.
b. $x^2 = 16y$, $(8, 4)$.
c. $y^2 + 5x = 0$, $(-5, 5)$.
d. $y = 6x^2$, $(1, 6)$.

10. Find the points at which the tangent to the corresponding curve in Problem 9 has the indicated slope:

a. $\frac{3}{4}$. *b.* $\frac{1}{2}$. *c.* $\frac{5}{6}$. *d.* 6.

11. Show that the line from the focus of the parabola $y^2 = 24x$ to the point where the tangent to the parabola at the point $(24, 24)$ cuts the y-axis is perpendicular to the tangent.

12. Find the angles of intersection of the graphs of the following pairs of equations:

a. $4y = x^2$, $2x = y$.
b. $y = x^2$, $x^2 + y^2 = 12$.
c. $x^2 = 4y$, $y = \dfrac{8}{x^2 + 4}$.

13. Find the tangent of the angle of intersection of the parabolas $y^2 = 2px$ and $x^2 = 2py$ at their point of intersection in the first quadrant.

14. What are the dimensions of the largest rectangle which can be inscribed in the segment of the parabola $x^2 = 24y$ cut off by the line $y = 6$?

15. Show that for any point on the parabola $y^2 = 2px$ the segment of the tangent between the point of tangency and the x-axis has a projection upon the x-axis of $2x$. From this deduce a geometric method of drawing a tangent to a parabola at any point.

16. Show that for any point on the parabola $y^2 = 2px$ the segment of the normal between the point of tangency and the x-axis has a projection upon the x-axis of length p.

17. Prove that the tangents to a parabola at the ends of the latus rectum

a. Are perpendicular. *b.* Intersect on the directrix.

18. Show that the equation of the tangent to the parabola $y^2 = 2px$ at the point (x_1, y_1) is $y_1 y = p(x + x_1)$.

103. Applications of the Parabola: the Parabolic Reflector. The parabola is frequently met with in the applications of mathematics to the sciences. The following are examples: the paths of some comets seem to be parabolic; arches are sometimes made in this shape; the first approximation to the path of a projectile is the parabola; the cable of a suspension bridge has the form of a parabola.

Among the most interesting of the applications is the parabolic reflector. The inner surface of such a reflector is that generated by rotating a parabola about its axis; the lamp is placed at the focus. The reason for its use is that every ray of light from the lamp at the focus is

reflected along a line parallel to the axis of the parabola. We will now prove this fact.

The figure represents a sectional view of the reflector. Let FP be any ray of light from the focus, which is reflected at P along the line PQ. Let CPT be tangent to the parabola at P. We have to prove that PQ is parallel to the axis CX for any position of P.

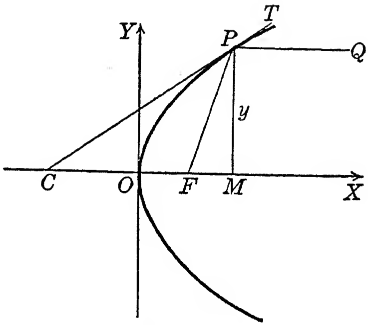


FIG. 101

By differentiating the equation of the parabola we find that its slope at P is $\frac{p}{y} = \tan PCX$. On the other hand, the figure shows that $\tan PCX = \frac{y}{CM}$. Hence

$$\frac{y}{CM} = \frac{p}{y}, \quad \text{or} \quad CM = \frac{y^2}{p} = \frac{2\,px}{p} = 2\,x.$$

But $\qquad OM = x,$

and therefore $\qquad CO = x;$

whence $\qquad CF = x + \frac{p}{2}.$

By the distance formula,

$$FP = \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2}$$

$$= \sqrt{x^2 - px + \frac{p^2}{4} + 2\,px}$$

$$= x + \frac{p}{2}.$$

We have thus shown that $FP = CF$, or that the triangle CFP is isosceles.

Therefore $\qquad$ angle $PCX =$ angle CPF.

But it is known from physics that if a ray of light meets a surface it is reflected in a line making equal angles with the tangent; that is,

$$\text{angle } CPF = \text{angle } TPQ.$$

Hence $$\text{angle } PCX = \text{angle } TPQ,$$

and the line PQ is parallel to CX by a well-known theorem of plane geometry. Thus all reflected rays have the direction of the axis of the parabola.

104. The Parabolic Arch. In practical problems involving the construction of a parabolic arch the dimensions given are the span ($AB = 2a$, in the figure) and the height ($CO = h$, in the figure).

The curve is then constructed in the following way: The rectangle $OCBD$ is drawn and OD is divided into a certain number (four in the figure) of equal parts by the points of division K_1, K_2, K_3. The side DB is divided into the same number of equal parts by the points L_1, L_2, L_3. Lines $K_1K'_1$, $K_2K'_2$, $K_3K'_3$ are drawn parallel to OC, and OL_1, OL_2, OL_3 are drawn. The intersections of OL_1 and $K_1K'_1$, OL_2 and $K_2K'_2$, etc. are points on the curve.

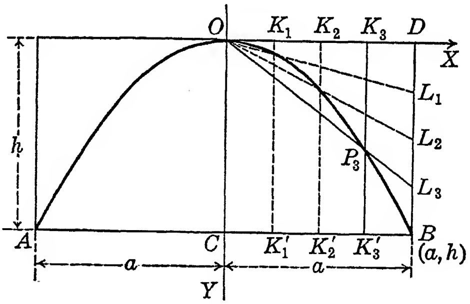


FIG. 102

The proof of the validity of this process consists in showing that the coördinates of the points so constructed will satisfy the equation of the curve. To find this equation it is customary to take the positive direction of the y-axis

as downward. Then the coördinates of the point B are (a, h). Since the parabola is symmetrical with respect to the y-axis, its equation will have the form $x^2 = 2py$. Since B lies on the curve, $a^2 = 2ph$; whence $2p = \frac{a^2}{h}$ and the equation of the curve is

$$hx^2 = a^2y. \tag{1}$$

To show that the points obtained in the figure satisfy equation (1), let us fix our attention on the point $P_3(x, y)$ of the figure. From similar triangles,

$$\frac{OK_3}{K_3P_3} = \frac{OD}{DL_3}; \quad \text{that is,} \quad \frac{x}{y} = \frac{a}{DL_3},$$

or

$$x(DL_3) = ay. \tag{2}$$

Since OD and DB were divided into the same number of equal parts,

$$\frac{OK_3}{DL_3} = \frac{OD}{DB}; \quad \text{that is,} \quad \frac{x}{DL_3} = \frac{a}{h},$$

or

$$xh = a(DL_3). \tag{3}$$

Eliminating (DL_3) from equations (2) and (3) gives

$$hx^2 = a^2y.$$

105. Other Equations of the Parabola. The equations of the parabola so far given are valid for a very restricted case; namely, that in which the vertex is at the origin and the directrix is parallel to one of the coördinate axes. If these conditions are not fulfilled, the equation is more complicated. In books on analytic geometry it is shown that this equation is always of the second degree, and transformations are given which enable us to find the vertex, the focus, etc.

These we shall not consider; for in problems where the focus and directrix of the parabola are known, it is usually possible to choose the origin and the axes in such a manner that we can use the simpler forms already derived.

On the other hand, it is frequently convenient to recognize the locus of the equation $y = a + bx + cx^2$ as a parabola with its axis perpendicular to the x-axis. Similarly, $x = a + by + cy^2$ is the equation of a parabola with its axis parallel to the x-axis. For a proof of these statements the student is referred to books on analytic geometry. He should, however, note that if $a = b = 0$ these forms are equivalent to those already discussed.

106. Discussion of the Equation $y = a + bx + cx^2$. For very large values of x the value of y will depend chiefly on the term cx^2. Hence if c is positive, y will be positive for large values of x, which means that the branches of the parabola extend upward. If c is negative, y will be negative for very large values of x, which means that the branches extend downward.

Differentiating, we find

$$y' = b + 2\,cx.$$

Setting $y' = 0$, we find $x = -\dfrac{b}{2\,c}\cdot$ Hence the point where $x = -\dfrac{b}{2\,c}$ is a maximum or minimum point and must therefore be the vertex. The axis is then $x = -\dfrac{b}{2\,c}\cdot$

No attempt should be made to memorize these results, but each equation of the type given should be treated in the above manner.

Example 1. Find the vertex and sketch the parabola whose equation is $4\,y = 4 + 8\,x - x^2$.

Solution. Since the coefficient of x^2 is negative, the branches of the parabola extend downward.

Differentiating, $y' = \dfrac{8 - 2x}{4}$.

Setting $y' = 0$, $x = 4$. The substitution of this value in the original equation gives $y = 5$. Hence the vertex is (4, 5) and the axis is $x = 4$.

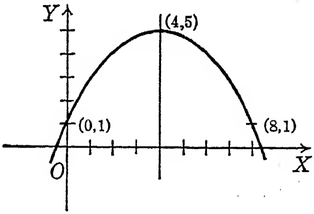


FIG. 103

A few more points are easily computed from the given equation. For example, $x = 0$ and $x = 8$ give the symmetrical points (0, 1) and (8, 1).

EXAMPLE 2. Find the equation of a parabola with a vertical axis and passing through the points (8, 1), (4, − 3), and (2, − 2).

Solution. Substituting these coördinates in the equation

$$y = a + bx + cx^2,$$

we obtain

$$1 = a + 8b + 64c,$$
$$-3 = a + 4b + 16c,$$
$$-2 = a + 2b + 4c.$$

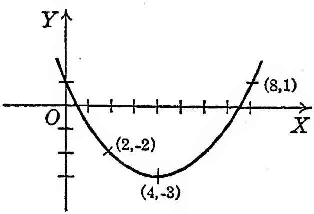


FIG. 104

Solving these equations simultaneously, we have $a = 1$, $b = -2$, and $c = \frac{1}{4}$. The desired equation is then

$$y = 1 - 2x + \tfrac{1}{4}x^2,$$

which reduces to $\quad 4y = 4 - 8x + x^2.$

PROBLEMS

1. Find the vertex of each of the following parabolas and draw the graph in each case:

a. $y = 4x - x^2$.

b. $2y = 8x - x^2$.

c. $4y = 8 + 16x - x^2$.

d. $3y = 4x^2 - 3$.

e. $5y = x^2 + 4x - 6$.

f. $4x = 9y^2 - 18y - 2$.

2. Find the equation of a parabola with a vertical axis through each of the following sets of points:

a. $(2, 5), (3, 0), (-7, 0)$.
b. $(5, -8), (2, -6), (-5, -3)$.
c. $(3, 11), (5, 3), (4, 5)$.
d. $(-2, 5), (-8, -4), (2, 4)$.
e. $(-1, 2), (-2, -3), (3, -1)$.
f. $(1, 6), (2, 4), (4, -2)$.

3. Find the equation of a parabola with its axis parallel to the x-axis and passing through each of the sets of points in Problem 2.

4. Using the definition of a parabola, find the equations of the parabolas satisfying the following data and draw their graphs:

a. Directrix $y = 7$, focus $(2, -1)$.
b. Directrix $y = -2$, focus $(4, 4)$.
c. Directrix $x = 6$, focus $(-2, 3)$.
d. Directrix $x = -4$, focus $(6, 2)$.

5. Plot carefully the graph of the parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$. What is its axis of symmetry? Where is its vertex?

6. At what point on the curve will the tangent to the parabola $y = 4 + 2x - x^2$ be parallel to the line $2x + y - 6 = 0$?

7. At what point on the curve will the tangent to the parabola $y = x^2 - 7x + 3$ be perpendicular to the line $x + 3y = 3$?

8. The path of a projectile from a mortar cannon lies on the parabola $y = 2x - x^2$; the unit is 1 mi., OX being horizontal and OY vertical, and the origin is the point of projection.

a. Find the direction of motion of the projectile at the instant of projection.

b. Find the direction of motion of the projectile when it strikes a vertical cliff $1\frac{1}{2}$ mi. distant.

c. Where will the path have an inclination of 45° to the horizontal?

d. What will be the highest point reached by the projectile?

9. Find the angle at which the parabolas $y = 4 - x^2$ and $y = x^2 - 14$ intersect.

10. Find the equation of the tangent to the parabola whose equation is $y = 4x - x^2$ which is parallel to the line joining the points (1, 3) and (4, 0).

11. In Problem 10 find the angles of intersection of the line and the parabola.

12. Find the points on the parabola whose equation is $y = 8x - x^2$ which are nearest to the point (4, 0).

13. The area of a cross section of a sphere of radius r, at a distance h from the surface, is given by the formula $A = 2\pi rh - \pi h^2$. Draw the graph of this function. Find the value of h, making A a maximum.

14. The melting-point t (degrees centigrade) of an alloy of lead and zinc is found to be

$$t = 133 + 0.875x + 0.01125x^2,$$

where x is the percentage of lead in the alloy. Draw the graph of this function. Is there a minimum value of t? How can it be found?

15. The cable of a suspension bridge assumes the shape of a parabola if the weight of the suspended roadbed (together with that of the cable) is uniformly distributed horizontally. Suppose that the towers of a bridge are 240 ft. apart and 60 ft. high and that the lowest point of the cable is 20 ft. above the roadway; find the vertical distance from the roadway to the cables at intervals of 20 ft.

THE ELLIPSE

107. Definitions. *The locus of a point the sum of whose distances from two fixed points is constant is an ellipse.*

The fixed points are called the *foci*, and the distance between the foci is denoted by $2c$.

The sum of the distances of any point of the ellipse from the foci is denoted by $2a$. Obviously $2a > 2c$, or $a > c$.

108. Construction of an Ellipse. The definition suggests at once a simple mechanical construction of an ellipse. Let pins be placed at the foci F and F', and a loop of string $F'PFF'$ of length $F'F + 2a$ be placed over the pins. If a pencil is placed at P and moved so as to keep the string taut, it will describe an ellipse. For then $F'P + FP$ will constantly equal $2a$.

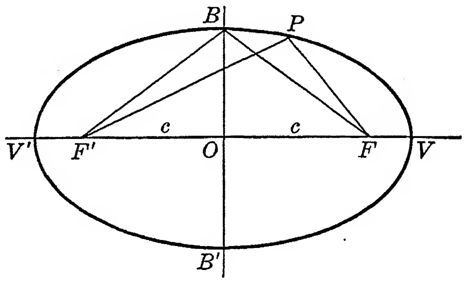


FIG. 105

109. Some Properties of the Ellipse. The preceding construction shows intuitively (as will be proved later analytically) that the curve is symmetrical with respect to the indefinite line through F and F', and also that it is symmetrical with respect to the line BB', which is the perpendicular bisector of FF'. The indefinite line through the foci is called the *principal axis* of the ellipse. The points V and V', where the principal axis cuts the curve, are the *vertices* of the ellipse.

Since the curve is symmetrical with respect to each of two perpendicular axes, it is symmetrical with respect to their point of intersection. Consequently the point O midway between the foci is called the *center* of the ellipse.

The distance OV from the center to one of the vertices is a.

Proof. $$F'V + FV = 2a.$$

But $$F'V = c + OV \quad \text{and} \quad FV = OV - c.$$

Substituting, $$c + OV + OV - c = 2a,$$

or $$2OV = 2a;$$

whence $$OV = a.$$

The point B is equidistant from F and F'. Hence $FB = a$. Let $OB = b$. Then, from the right triangle OBF, $b^2 = a^2 - c^2$. Hence $b < a$; that is, $OB < OV$ and $B'B < V'V$. For this reason the axis $V'V (= 2a)$ is called the *major axis* of the ellipse, and the axis $B'B (= 2b)$ is called the *minor axis*.

110. Equation of the Ellipse. Let the foci be on the x-axis, with the origin midway between them. Then their coördinates are $(c, 0)$ and $(-c, 0)$.

If $P(x, y)$ is any point of the ellipse, the definition requires that $F'P + FP = 2a$. Hence, by the distance formula,

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a.$$

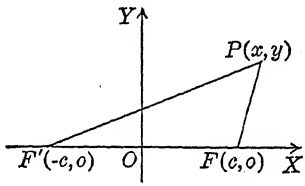


FIG. 106

Transposing the second of these radicals, squaring, and collecting terms, we have

$$4cx - 4a^2 = -4a\sqrt{(x-c)^2+y^2}.$$

Dividing by 4 and squaring again, we get

$$c^2x^2 - 2a^2cx + a^4 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2,$$

or $$(a^2 - c^2)x^2 + a^2y^2 = a^4 - a^2c^2 = a^2(a^2 - c^2).$$

But $$a^2 - c^2 = b^2.$$

This gives $$b^2x^2 + a^2y^2 = a^2b^2.$$

Dividing both sides by a^2b^2, we obtain the standard equation

(IV) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

111. Discussion of the Ellipse. The discussion of equation IV verifies the properties of the ellipse stated in § 109.

Intercepts. Setting $y = 0$ and $x = 0$ successively, we find that the x-intercepts are $\pm a$ and that the y-intercepts are $\pm b$.

Symmetry. Obviously there is symmetry with respect to both axes; hence the origin is a center of symmetry.

Extent. The terms $\frac{x^2}{a^2}$ and $\frac{y^2}{b^2}$ are both squares and hence never negative. Since their sum is constantly equal to 1, two facts are at once apparent: (1) As x increases numerically y decreases numerically, and vice versa. (2) Neither term can be greater than 1; hence x is never greater numerically than a, and y is never greater numerically than b.

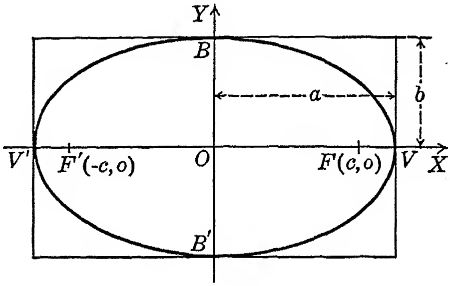


Fig. 107

Thus the ellipse lies wholly within the rectangle bounded by the lines $x = \pm a$ and $y = \pm b$.

112. Equation of Ellipse with Foci on the Y-axis. If the foci are on the y-axis and the center is at the origin, the coördinates of the foci are $(0, c)$ and $(0, -c)$. Taking $F'P + FP = 2a$, and proceeding as in § 110, we obtain the equation

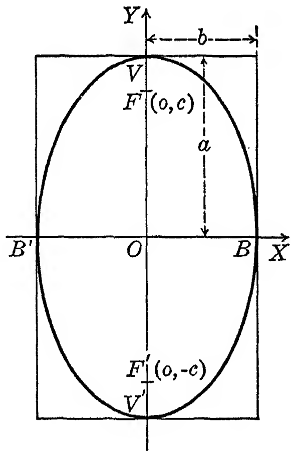


Fig. 108

$$\text{(IV}a) \qquad \frac{y^2}{a^2} + \frac{x^2}{b^2} = 1.$$

In this case the coördinates of the vertices are $(0, \pm a)$, and those of the ends of the minor axis are $(\pm b, 0)$.

113. Sketching the Ellipse. Inspection of the standard equations IV and IV a shows that the locus of any equation of the form $Ax^2 + Cy^2 = F$, where A, C, and F are positive, is an ellipse with its center at the origin and with its foci on one of the coördinate axes. For such an equation can be reduced to one of the two standard forms by dividing through by F and writing the equation in the form

$$\frac{x^2}{\frac{F}{A}} + \frac{y^2}{\frac{F}{C}} = 1.$$

To sketch the locus of such an equation we first find the intercepts. The larger intercept is half the major axis (semi-major axis), and the smaller intercept is half the minor axis (semi-minor axis). To avoid the tendency to make the ellipse too pointed, compute the coördinates of an additional pair of points near each vertex.

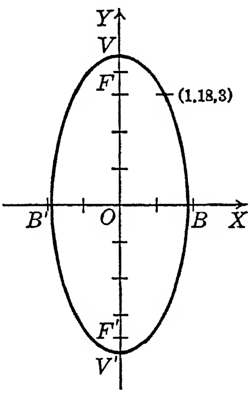


FIG. 109

EXAMPLE. Sketch the ellipse whose equation is $5x^2 + y^2 = 16$. Also find its foci.

Solution. The intercepts on the x-axis are $\pm\sqrt{\frac{16}{5}} = \pm 1.79$. The intercepts on the y-axis are ± 4. The major axis lies along the y-axis. Hence $a = 4$ and $b = \sqrt{\frac{16}{5}} = 1.79$. From the relation $c^2 = a^2 - b^2$ we find that $c = \sqrt{\frac{64}{5}} = 3.58$. The foci are then $(0, \pm 3.58)$. Four additional points are obtained by taking $y = \pm 3$, giving $x = \pm\sqrt{\frac{7}{5}} = \pm 1.18$.

114. The Circle as a Limiting Form of the Ellipse. If we take $b = a$, the standard equation of the ellipse reduces to $x^2 + y^2 = a^2$, which is the equation of a circle of radius a.

Thus the circle is a special form of the ellipse. It is instructive to see just how this happens.

If in the equation $b^2 = a^2 - c^2$ we take $b = a$, c becomes 0. Hence a circle is an ellipse whose foci coincide. This is also obvious from a consideration of the construction explained in § 108.

PROBLEMS

1. Find the major and minor axes and the coördinates of the foci of the following ellipses, and sketch the curves:

a. $x^2 + 4y^2 = 16$.
b. $x^2 + 9y^2 = 45$.
c. $16x^2 + y^2 = 64$.
d. $16x^2 + 9y^2 = 144$.
e. $x^2 + 5y^2 = 10$.
f. $3x^2 + y^2 = 9$.
g. $25x^2 + 4y^2 = 25$.
h. $2x^2 + 3y^2 = 36$.
i. $5x^2 + y^2 = 10$.
j. $9x^2 + 4y^2 = 16$.
k. $4x^2 + 25y^2 = 100$.
l. $4x^2 + 25y^2 = 1$.
m. $10x^2 + 20y^2 = 49$.
n. $9x^2 + 4y^2 = 50$.

2. Find the equation of the ellipse whose center is at the origin and which satisfies the following conditions:

a. x-intercepts ± 14, y-intercepts ± 7.
b. Major axis 20, minor axis 12, vertices on the x-axis.
c. Major axis 20, minor axis 12, vertices on the y-axis.
d. Major axis 16, minor axis 8, vertices on the x-axis.
e. Major axis 16, minor axis 8, vertices on the y-axis.

3. Find the equation of the ellipse with center at the origin and draw the curve, given

a. One vertex (5, 0), one focus (3, 0).
b. One focus (4, 0), minor axis 8.
c. One vertex (0, 5), minor axis 6.
d. One vertex (0, 6), one focus (0, 4).

4. Find the equation of the ellipse with center at the origin and with principal axis along one of the coördinate axes, and passing through

a. (4, 3) and (6, 2).
b. (−1, 6) and (2, 3).

5. Are the following points on, inside, or outside the ellipse $x^2 + 4y^2 = 16$?

a. $(2, \frac{9}{5})$. *b.* $(\frac{17}{5}, -1)$. *c.* $(\frac{3}{2}, \frac{7}{4})$.

6. The chord of an ellipse through the focus perpendicular to the principal axis is called the latus rectum. Prove that its length is $\frac{2b^2}{a}$.

7. From points on the circumference of the circle whose equation is $x^2 + y^2 = 64$, perpendiculars are drawn to the x-axis. Find the equation of the locus of a point which moves so as to bisect each perpendicular.

8. Find the equation of the locus of the vertex of a triangle whose base is the line joining $(a, 0)$ and $(-a, 0)$ and in which the product of the tangents of the base angles is $\frac{b^2}{a^2}$.

9. Show that the slope of the line which is tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at any point is $-\frac{b^2x}{a^2y}$.

10. Find the equations of the tangent and normal to the ellipse $x^2 + 3y^2 = 21$ at the point in the fourth quadrant where $x = 3$.

11. Find the equations of the two tangents to the ellipse $x^2 + 4y^2 = 18$ which pass through the point $(2, 2)$.

12. A circle is circumscribed about the ellipse whose equation is $4x^2 + y^2 = 16$, and a tangent is drawn to each curve in the first quadrant at the point whose ordinate is 2. Find the point of intersection of these tangents.

13. Find the angles of intersection of the following curves:

a. $x^2 + 4y^2 = 8$, $x^2 = 2y + 2$.
b. $9x^2 + 4y^2 = 36$, $4x^2 + 9y^2 = 36$.
c. $2x^2 + y^2 = 24$, $x + 2y = 10$.
d. $4x^2 + y^2 = 8$, $y^2 = 4x$.
e. $x^2 + 4y^2 = 65$, $y = 4x$.

14. At what point on the ellipse $16x^2 + 9y^2 = 400$ does y decrease at the same rate that x increases?

15. A point generates the curve $x^2 + 4y^2 = 20$ and moves with the x-component of its velocity constantly equal to -2 units/sec. Find its speed along the curve and its direction of motion when $y = 2$.

16. The lower base of an isosceles trapezoid is the major axis of an ellipse; the ends of the upper base are points on the ellipse. Show that the maximum trapezoid of this type has the length of its upper base half that of the lower.

17. Find the area of the largest rectangle that can be inscribed in an ellipse.

18. An isosceles triangle is to be inscribed in the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, the vertex being taken at $(0, b)$. Find the equation of the base if the triangle is a maximum.

19. Using the result of Problem 9 and the formula for the angle between two lines, show that the acute angle between a tangent at any point and the line joining this point to either focus is given by the formula

$$\tan\theta = \frac{b^2}{cy}.$$

20. Prove that the equation of the tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at the point (x_1, y_1), which lies on the ellipse, is $b^2x_1x + a^2y_1y = a^2b^2$.

THE HYPERBOLA

115. Definitions. *The locus of a point the difference of whose distances from two fixed points is constant is a hyperbola.*

The fixed points are called the *foci*, and the distance between the foci is denoted by $2c$.

The difference of the distances of any point of the hyperbola from the foci is denoted by $2a$. By a theorem in plane geometry, $2a < 2c$, or $a < c$.

116. Construction of a Hyperbola. Let F and F' be the foci, and suppose one end of a string, indefinite in length, is passed through a hole in the paper at F, while the other end is passed through a hole at F'. When the string is drawn tight, suppose a pencil is knotted in it at V so that $F'V - FV = 2a$. Then let the string be drawn out through each hole *at the same rate* while the pencil is moved so as to keep the two parts taut. Then, for any position P of the pencil, $F'P - FP = 2a$, and, by definition, the pencil will describe a hyperbola. The definition does not prescribe the order in which the difference of the distances shall be taken. If the pencil is knotted at V' so that $FV' - F'V' = 2a$, the same process will generate another curve, $QV'Q$. These two curves together constitute the hyperbola, each curve being called a *branch* of the hyperbola.

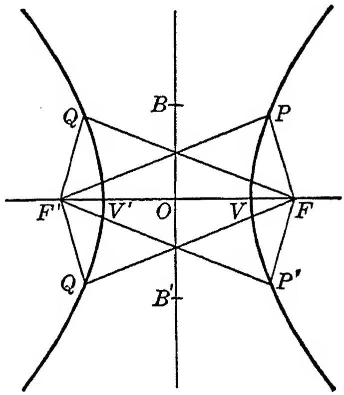


FIG. 110

117. Some Properties of the Hyperbola. The preceding construction shows intuitively (as will be proved later analytically) that the hyperbola is symmetrical with respect to the indefinite line through F and F' and also that it is symmetrical with respect to the line BB', which is the perpendicular bisector of FF'. The indefinite line through the foci is called the *principal axis* of the hyperbola. The points V and V', where the principal axis cuts the curve, are the vertices of the hyperbola.

Since the curve is symmetrical with respect to the perpendicular lines FF' and BB', it is symmetrical with respect

to their point of intersection. The point O midway between the foci is called the *center* of the hyperbola.

The distance OV from the center to one of the vertices is a. The proof is similar to that in § 109.

The segment of the principal axis VV' $(= 2a)$ is called the *transverse* axis of the hyperbola.

The points B and B' are each at a distance b from O, where $b = \sqrt{c^2 - a^2}$. The segment BB' $(= 2b)$ is called the *conjugate* axis of the hyperbola. Note that since, by definition, $b^2 = c^2 - a^2$, whence $c^2 = a^2 + b^2$, b may be either larger or smaller than a.

118. Equation of the Hyperbola. Let the foci be on the x-axis, with the origin midway between them. Then their coördinates are $(c, 0)$ and $(-c, 0)$.

If $P(x, y)$ is any point on the hyperbola, the definition requires that

$$F'P - FP = \pm 2a.$$

Hence, by the distance formula,

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a.$$

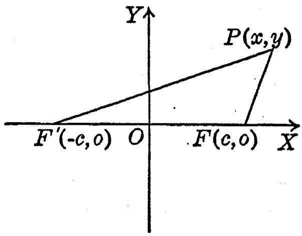


FIG. 111

Transposing the second of these radicals, squaring, and collecting terms, we have

$$cx - a^2 = \pm a\sqrt{(x-c)^2 + y^2}.$$

If we square this again, it may be reduced to

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2).$$

Since $b^2 = c^2 - a^2$, this gives

$$b^2x^2 - a^2y^2 = a^2b^2.$$

Dividing by a^2b^2, we obtain the standard equation

$$\text{(V)} \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The student should compare the above carefully with § 110.

119. Discussion of the Equation. *Intercepts.* Setting $y=0$, the x-intercepts are seen to be $\pm a$. Setting $x=0$, we find that the y-intercepts are imaginary. Hence the curve does not cross the y-axis, and, since there are points of intersection with the x-axis on both sides of the origin, the curve must consist of two *branches.*

Symmetry. Obviously there is symmetry with respect to both axes and the origin.

Extent. If we write equation V in the form $\frac{x^2}{a^2}=1+\frac{y^2}{b^2}$, we see at once that $\frac{x^2}{a^2}$ is never less than 1, and hence that x is always numerically greater than or equal to a. Also, as y increases, x increases, numerically. On the other hand, y may have any value.

Therefore the two branches of the hyperbola lie outside the lines $x=\pm a$ and recede to infinity from both axes in all four quadrants.

120. Asymptotes. If the equation is solved for y in terms of x, we obtain $y=\pm\frac{b}{a}\sqrt{x^2-a^2}$, a result which indicates that, for values of x large in comparison with that of a, y is approximated by $\pm\frac{bx}{a}$. This suggests that the hyperbola approaches the lines whose equations are $y=\pm\frac{bx}{a}$ as it recedes to infinity. Lines approached by a curve in this manner are called *asymptotes.*

To prove the above statements, take a point $P(x, y)$ on the hyperbola in the first quadrant and let $P_1(x, y_1)$ be the point on the line $y=\frac{bx}{a}$ which has the same abscissa. We must show that as x approaches infinity y_1-y approaches 0.

From the equations of the line and of the hyperbola, we have

$$y_1 - y = \frac{bx}{a} - \frac{b\sqrt{x^2 - a^2}}{a}$$

$$= \frac{b(x - \sqrt{x^2 - a^2})}{a}.$$

Multiplying numerator and denominator by $x + \sqrt{x^2 - a^2}$, we have

$$y_1 - y = \frac{ab}{x + \sqrt{x^2 - a^2}}.$$

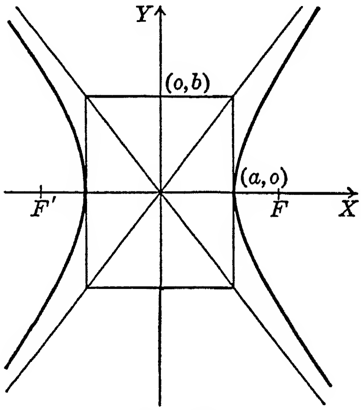


FIG. 112

If x approaches infinity the denominator obviously approaches infinity, while the numerator remains constant; hence

$$y_1 - y \rightarrow 0.$$

The other quadrants are treated in the same way.

We have thus shown that as the hyperbola recedes to infinity it approaches asymptotically lines through the origin with slopes $\pm \frac{b}{a}$. These are easily seen to be the diagonals of the rectangle whose sides are the lines $x = \pm a$ and $y = \pm b$.

121. Equation of Hyperbola with Foci on the Y-axis. Let us take the foci at $(0, c)$ and $(0, -c)$. Proceeding exactly as in § 118, we obtain a second standard form,

$$\text{(V }a\text{)} \qquad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

For this form the vertices are $(0, \pm a)$. No value of y is numerically less than a, and x may have any value. Solving for y,

$$y = \pm \frac{a}{b}\sqrt{x^2 + b^2}.$$

From this it may be shown, as in § 120, that for this form the asymptotes are the lines

$$y = \pm \frac{ax}{b}.$$

These asymptotes are the diagonals of the rectangle whose sides are the lines $x = \pm b$, $y = \pm a$.

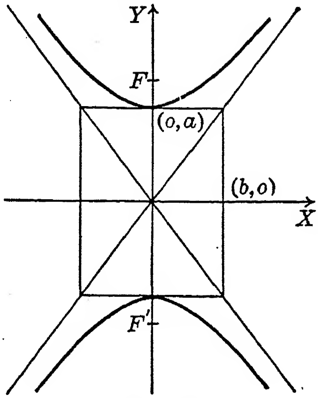


FIG. 113

122. Sketching the Hyperbola. Inspection of the standard equations V and Va shows that the locus of any equation of the form $Ax^2 - Cy^2 = \pm F$, where A, C, and F are positive, is a hyperbola with its center at the origin and with the foci on one of the coördinate axes. For such an equation can be reduced to one of the two standard forms by dividing through by F and writing the equation in the form

$$\frac{x^2}{\frac{F}{A}} - \frac{y^2}{\frac{F}{C}} = \pm 1.$$

To sketch the locus of such an equation we first find the intercepts, which will be real on one of the coördinate axes and imaginary on the other. The real intercept is half the transverse axis (*semi-transverse axis*). The foci and vertices lie upon the coördinate axis on which the intercepts are real. A sketch of the curve may be made by proceeding as in the example on the following page.

EXAMPLE. Find the vertices and foci of the hyperbola $4x^2 - 9y^2 = 36$, and draw the curve.

Solution. The real intercepts are $x = \pm 3$, and the imaginary intercepts are $y = \pm 2\sqrt{-1}$. Thus $a = 3$, $b = 2$, and $c = \sqrt{13} = 3.61$. Hence the vertices are $(\pm 3, 0)$ and the foci are $(\pm 3.61, 0)$.

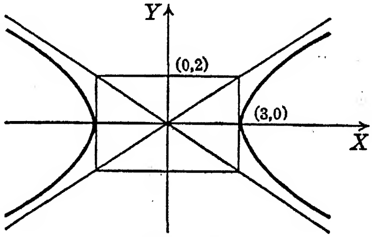


FIG. 114

Four other points are obtained by taking $x = \pm 4$, whence $y = \pm 1.76$. The slopes of the asymptotes are $\pm \frac{b}{a} = \pm \frac{2}{3}$, and they may be drawn as the diagonals of the rectangle whose sides are $x = \pm 3$ and $y = \pm 2$. Using the asymptotes as guiding lines, we draw each branch of the curve through the three points plotted.

PROBLEMS

1. Sketch the following hyperbolas with their asymptotes. Find the foci in each case.

a. $x^2 - y^2 - 4 = 0$.
b. $x^2 - y^2 + 4 = 0$.
c. $x^2 - 4y^2 - 16 = 0$.
d. $4x^2 - y^2 - 16 = 0$.
e. $4x^2 - y^2 + 25 = 0$.
f. $9x^2 - 16y^2 + 144 = 0$.
g. $2x^2 - y^2 - 8 = 0$.
h. $x^2 - 2y^2 + 1 = 0$.
i. $5x^2 - 7y^2 - 35 = 0$.
j. $3x^2 - y^2 + 9 = 0$.
k. $6x^2 - 2y^2 + 25 = 0$.
l. $3x^2 - 8y^2 - 36 = 0$.

2. Find the equations of the hyperbolas satisfying the following conditions:

a. $a = 10$, $b = 5$, foci on the x-axis.
b. $a = 10$, $b = 5$, foci on the y-axis.
c. $a = 4$, $b = 8$, foci on the x-axis.
d. Vertex (3, 0), focus (4, 0).
e. Vertex (0, 2), focus (0, 5).

f. Vertex (4, 0), focus (6, 0).
g. Focus (2, 0), conjugate axis 2.
h. Focus (0, 5), conjugate axis 6.
i. Vertex (0, 4), conjugate axis 8.

3. Find the equation of the hyperbola having its center at the origin and its foci on the x-axis, and passing through

a. (6, 2) and $(2\sqrt{6}, 1)$. *b.* (2, 2) and (4, 5).

4. Find the equation of the locus of a point which moves so that the difference of its distances from the points $(0, \pm 13)$ is 10.

5. The latus rectum of a hyperbola is a chord through the focus perpendicular to the principal axis. From the equation show that the length of the latus rectum is $\dfrac{2b^2}{a}$.

6. Find the equations of the tangent and the normal to each of the following hyperbolas at the point indicated:

a. $x^2 - y^2 = 9$, (5, 4). *c.* $x^2 - y^2 + 9 = 0$, $(-4, 5)$.
b. $4x^2 - y^2 - 39 = 0$, $(4, -5)$. *d.* $x^2 - 4y^2 = 9$, $(-5, 2)$.

7. Find the equations of the tangents to the hyperbola $2x^2 - y^2 = 9$ which are parallel to the line $2x - y + 7 = 0$.

8. Find the equations of the tangents to the hyperbola $x^2 - y^2 + 16 = 0$ which are perpendicular to the line $5x + 3y - 15 = 0$.

9. Find the equations of the tangents to the hyperbola $x^2 - 8y^2 = 1$ which pass through the point $(-5, -2)$.

10. Find the angles of intersection of the line $2x - 3y = 0$ and the hyperbola $x^2 - y^2 = 5$.

11. Find the slope of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ in terms of x and y, and in terms of x alone. Show that the limit of the slope as x approaches infinity is $\pm\dfrac{b}{a}$.

12. Show that the equation of the tangent to the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ at the point (x_1, y_1), which lies on the curve, is $b^2x_1x - a^2y_1y = a^2b^2$.

13. A point moves along the hyperbola $x^2 - 4y^2 = 20$ in such a way that x is increasing at the rate of 3 units/sec. Find the rate at which y is changing when the point passes through $(6, -2)$.

14. A segment is bounded by the line $x = 8$ and the hyperbola $x^2 - y^2 = 16$. Find the dimensions of the largest rectangle which can be inscribed in this segment.

15. Find the point on the hyperbola $x^2 - y^2 - 16 = 0$ which is nearest to the point $(0, 6)$.

123. The Equilateral Hyperbola. If $a = b$, the standard forms of the equation reduce to

$$x^2 - y^2 = a^2$$

and

$$y^2 - x^2 = a^2.$$

These are called *equilateral* or *rectangular* hyperbolas. The latter name is due to the fact that the asymptotes have their slopes ± 1 and are therefore at right angles to each other.

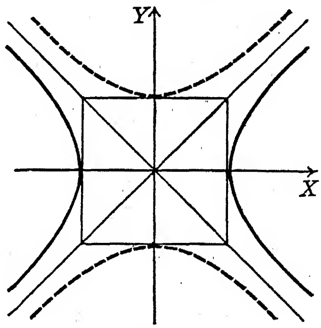


FIG. 115

124. Other Forms of the Equation of the Equilateral Hyperbola. It is shown in analytic geometry that the locus of the equation $xy = C$ is an equilateral hyperbola. The asymptotes are easily seen to be the x-axis and the y-axis. If C is negative the two branches are in the second and fourth quadrants. The vertices are on the line $y = x$ or $y = -x$.

A slightly more general form is $xy + ax + by + c = 0$. It may be shown that the asymptotes are $x = -b$ and $y = -a$.

The importance of these forms is due chiefly to the fact that many functional relations have equations of these types.

125. The Conics. The equations of the circle, the parabola, the ellipse, and the hyperbola, discussed in this chapter, are all special forms of the general quadratic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

It is shown in analytic geometry that the locus of this equation is always one of the curves mentioned, exception being made of certain degenerate forms. Furthermore, all these curves may be obtained as the intersection of a plane and a right circular conical surface. For this reason they are called *conic sections* or simply *conics.* The ellipse and hyperbola are called *central conics.*

PROBLEMS

1. Name and sketch each of the following curves:

a. $xy + 8 = 0.$

b. $x^2 + y^2 - 4 = 0.$

c. $x^2 - y^2 + 4 = 0.$

d. $x^2 + y - 4 = 0.$

e. $4x^2 + y^2 - 64 = 0.$

f. $x + 4y^2 - 64 = 0.$

g. $4x^2 - y^2 - 64 = 0.$

h. $4x^2 + y = 0.$

i. $4x^2 - y^2 = 0.$

j. $4x^2 + y^2 = 0.$

k. $xy - 6x + 7y - 42 = 0.$

l. $xy + 4x - 2y + 8 = 0.$

2. Two points are 2000 ft. apart. At one of these points the report of a cannon is heard one second later than at the other. By means of the definition of a hyperbola show that the cannon is somewhere on a certain hyperbola, and write its equation after making a suitable choice of axes. (The velocity of sound is 1090 ft./sec.)

3. Find the equation of the hyperbola through the point (1, 1), with asymptotes $y = \pm 2x$.

4. Find the slope of the hyperbola $xy + ax + by + c = 0$ at any point.

5. Show that at any point the slope of the hyperbola $xy = C$ is $-\dfrac{y}{x}$.

6. Find the point of the hyperbola $xy = 48$ where the slope is $-\frac{3}{4}$.

7. Find the equations of the tangent and normal to each of the following curves at the point indicated:

$a.$ $y^2 + 16x = 0$, $(-4, 8)$.
$b.$ $xy + 24 = 0$, $(4, -6)$.
$c.$ $4x^2 + y^2 = 100$, $(-3, 8)$.
$d.$ $x^2 + y^2 - 12x + 6y = 55$, $(-2, 3)$.
$e.$ $x^2 - 4y^2 + 64 = 0$, $(6, 5)$.

8. At the point $(5, \frac{9}{4})$ of the hyperbola $9x^2 - 16y^2 = 144$ a tangent and a normal are drawn. Find the area of the triangle bounded by the tangent, the normal, and the x-axis.

9. Show that the hyperbola $x^2 - y^2 = 5$ and the ellipse $4x^2 + 9y^2 = 72$ intersect at right angles.

10. Show that the hyperbolas $x^2 - y^2 = a^2$ and $xy = a^2\sqrt{2}$ intersect at right angles.

11. Find the angles of intersection of the following pairs of curves:

$a.$ $y = x^2$, $x^2 - y^2 + 12 = 0$.
$b.$ $x^2 + y^2 = 104$, $x^2 - 9y^2 = 64$.
$c.$ $xy = 16$, $x^2 + 4y^2 = 80$.
$d.$ $xy = -48$, $x^2 + y^2 = 100$.

12. Show that the tangent to the hyperbola $xy = C$ at any point forms with the coördinate axes a right triangle whose area is $2xy = 2C$.

13. Show that the equation of the tangent to the hyperbola $xy = C$ at the point (x_1, y_1), which lies on the curve, is $x_1y + y_1x = 2C$.

14. For what value of k will the line $y = x + k$ be tangent to the hyperbola $x^2 - 9y^2 = 72$?

15. Prove that the triangle formed by a tangent to the curve $2xy = a^2$, the x-axis, and the line joining the point of contact to the origin is isosceles.

HINT. Compare the slopes of the sides.

CHAPTER VIII

CURVE-TRACING

126. In previous chapters the student has learned how to plot curves by points and how to find their simpler properties — symmetry, slope, and maximum and minimum points. He has also studied the conics in some detail. It is now in order to consider curves of a more general nature and to study the methods applicable to their discussion.

The graph of any equation in x and y can be constructed by assuming values for x and computing the corresponding values of y, or vice versa. But this method alone is often laborious and ineffective. It is laborious because all facts about the graph are discovered by trial and because there is no advance information concerning the most important values of x (or y) to be chosen. It is ineffective because the most painstaking and extensive calculations may fail to show certain important properties of the curve.

In Chapter II we utilized the notions of intercepts and symmetry, for assistance in making tables of values. It was there stated that for a thorough discussion of a curve the methods of the calculus are necessary. Some of the methods have already been given; these will be amplified and others will be added in this chapter.

127. Tracing a Curve. To "trace a curve" is to draw the graph after having determined its most important properties. This always involves the computation of values of one variable corresponding to certain values of the other; but these values are carefully chosen so as to include those

that show the nature of the curve and to avoid the necessity of computing many values that have no special significance.

The "discussion of an equation" is the determination of the properties of the curve represented by the equation. The points to be considered are taken up in the following paragraphs.

128. Intercepts. The method of finding the intercepts of a curve was given in §§ 17, 28. The intercepts of a curve may not be of special importance in determining its shape, but the points determined by them are often the easiest points to find from the equation and frequently serve as a check on the work. After the intercepts have been found, we know that the curve can cross the coördinate axes at *no other points.*

129. Symmetry. The principle of symmetry has been explained in an earlier paragraph (§ 28). The rules for recognizing the symmetry of a curve from its equation are repeated for convenience.

1. A curve is symmetrical with respect to the x-axis if its equation contains only even powers of y, and conversely.
2. A curve is symmetrical with respect to the y-axis if its equation contains only even powers of x, and conversely.
3. A curve is symmetrical with respect to the origin if its equation remains unchanged after substituting $-x$ for x and $-y$ for y, and conversely.

130. Excluded Values. If when a value is substituted for x in the equation of a curve the resulting value of y involves a square root (or an even root) of a negative number, the value of y is imaginary and there is no corresponding point on the curve. If when $x=a$ all the corresponding values of y are imaginary, it means geometrically that the curve does not cross the line $x=a$. Hence, in computing values of y, we should exclude from the table

all values of x which will make y imaginary. Similarly, all values of y should be excluded which will make x imaginary.

To determine in advance what values of x are to be excluded, the equation of the curve must be solved for y algebraically. An inspection of the result will show what values of x, if any, will make y imaginary. Similarly, to determine what values of y should be excluded, the equation must be solved for x algebraically.

The determination of the values to be excluded shows the *extent* of the curve. For example, if it appears that all negative values of x must be excluded, there is no part of the curve to the left of the y-axis, and the curve extends indefinitely to the right.

131. Examples. In applying the previous and the following sections to specific problems, it is important to keep in mind the two purposes of a discussion: first, it facilitates the labor of computing the table of values; second, it serves as a check upon the table of values, and vice versa. It should be unnecessary to state that the curve as drawn must have the properties given in the discussion. If some of the points given by the computed table of values do not agree with the discussion, there is an error in either the computation or the discussion.

EXAMPLE 1. Given the curve whose equation is $y^2 = x + 4$.

a. Find the intercepts. *c.* Find the extent.
b. Discuss the symmetry. *d.* Draw the curve.

Solution. *a.* Setting $y = 0$, the x-intercept is found to be -4. Setting $x = 0$, the y-intercepts are found to be $+2$ and -2.

b. The curve is symmetrical with respect to the x-axis (see § 129). It is not symmetrical with respect to the y-axis or the origin.

c. Solving for y, $\qquad y = \pm\sqrt{x+4}.$

All values of x such that $x + 4$ is negative must be excluded. Hence $x < -4$ must be excluded.

Solving for x, $\quad x = y^2 - 4.$

No values of y need be excluded.

Hence the curve does not extend to the left of the line $x = -4$. It extends indefinitely to the right and indefinitely upward and downward.

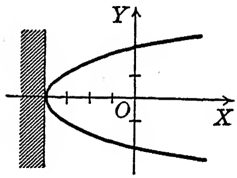


FIG. 116

d. The curve is drawn by plotting a few points in addition to the intercepts already found. The shaded area indicates the region of excluded values.

EXAMPLE 2. The equation of a curve is $x^2 + 4y^2 - 16y = 0$.

a. Find the intercepts. *c.* Find the extent.
b. Discuss the symmetry. *d.* Draw the curve.

Solution. *a.* When $y = 0$, $x = 0$; when $x = 0$, $y = 0$ or 4.

b. The tests for symmetry show symmetry with respect to the y-axis, but not with respect to the x-axis or the origin.

c. Solving the equation for y, we obtain

$$y = 2 \pm \frac{\sqrt{16 - x^2}}{2}.$$

Hence x^2 cannot be greater than 16; that is, x cannot be numerically greater than 4. This fact is expressed by writing $|x| > 4$ excluded.

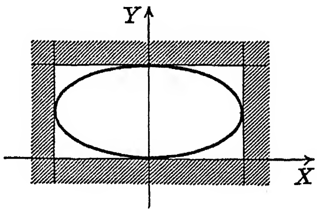


FIG. 117

Solving the equation for x, we obtain

$$x = \pm 2\sqrt{4y - y^2}$$
$$= \pm 2\sqrt{y(4 - y)}.$$

Hence we must exclude $y > 4$ and $y < 0$.

The excluded values show that the curve does not extend outside the rectangle bounded by the lines $x = 4$, $x = -4$, $y = 0$, and $y = 4$.

d. The curve is shown in the figure, the shaded area indicating the region of excluded values.

PROBLEMS

(*a*) Find the intercepts, (*b*) discuss the symmetry, (*c*) find the extent, and (*d*) draw the graph of the following equations:

1. $x^2 + 4y^2 - 6x = 91$.

2. $x^2 + y = 2$.

3. $x^3 + y^3 = 1$.

4. $y = x^3$ (*cubical parabola*).

5. $y^2 = x^3$ (*semicubical parabola*).

6. $y = x^3 - 4x$.

7. $y^2 = x^3 - 4x$.

8. $x^2y = 4$.

9. $x^2y^2 = 4$.

10. $y^3 = x^2$ (*semicubical parabola*).

11. $x^4 + y^4 = a^4$.

12. $x^2 + y^2 + 6x + 8y = 0$.

13. $9x^2 + 25y^2 - 54x - 144 = 0$.

14. $x^2 - y^2 + 4x + 3 = 0$.

15. $y = \dfrac{8a^3}{x^2 + 4a^2}$ (*witch*).

16. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ (*hypocycloid of four cusps*).

17. $y = 2^{-x^2}$.

18. $y^2(2a - x) = x^3$ (*cissoid*).

19. $\sqrt{x} + \sqrt{y} = \sqrt{a}$ (*parabola*).

20. $y^2 = \dfrac{x^2(a + x)}{(a - x)}$ (*strophoid*).

21. $(y - x^2)^2 = x^5$.

22. $y = x^4 - 4x^2$.

23. $y^2 = 4x^2 - x^4$.

24. $y = \dfrac{x}{x^2 + 1}$.

132. Asymptotes. *An asymptote is a line approached by a curve as it recedes to infinity.* We shall here consider only vertical and horizontal asymptotes.

Consider the equation $xy - y = 1$. Solving for y, we have

$$y = \frac{1}{x-1},$$

from which it appears that y increases indefinitely as x approaches the value 1. The curve is said to approach the line $x = 1$ *asymptotically*, and the line $x = 1$ is a *vertical asymptote* of the curve.

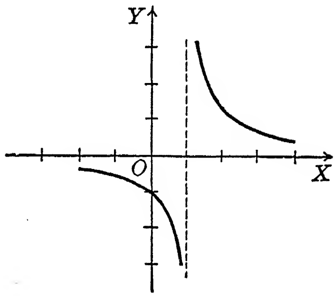


FIG. 118

Similarly, by solving for x,

$$x = \frac{y+1}{y},$$

from which we see that the curve approaches the x-axis asymptotically and the x-axis is a *horizontal asymptote* of the curve.

In general the line $x = a$ will be a vertical asymptote of a curve if, from the equation of the curve, y becomes infinite when x approaches a. Similarly, the line $y = b$ will be a horizontal asymptote of a curve if x becomes infinite when y approaches b.

Since we are considering only algebraic equations, y can become infinite when x approaches a only if the expression for y is a fraction such that the denominator is zero when $x = a$. Hence, to find the vertical asymptotes of a curve whose equation is given we solve the equation for y. If the result is a fraction, we set the denominator equal to zero and solve for x. The real roots will give the vertical asymptotes. If there are no real roots, the curve has no vertical asymptotes.

Similarly, to find the horizontal asymptotes of a curve whose equation is given we solve the equation for x. If

the result is a fraction, we set the denominator equal to zero and solve for y. The real roots will give the horizontal asymptotes.

Example. Find the asymptotes of the curve whose equation is

$$x^2y - y - 2x^2 + x = 0$$

and plot the curve.

Solution. Solving the equation for y, we obtain

$$y = \frac{2x^2 - x}{x^2 - 1}.$$

Setting the denominator equal to zero and solving gives $x = \pm 1$. Hence there are two vertical asymptotes: $x = -1$ and $x = +1$.

To solve the equation for x write it in the form

$$(y - 2)x^2 + x - y = 0$$

and use the quadratic formula. We obtain

$$x = \frac{-1 \pm \sqrt{1 - 8y + 4y^2}}{2(y - 2)}.$$

Hence the horizontal asymptote is $y = 2$.

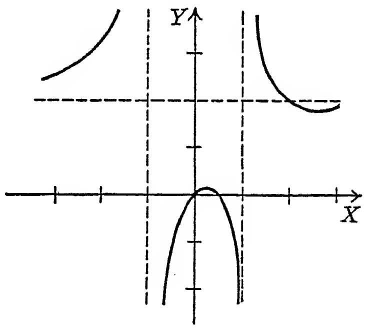


Fig. 119

In making the table of values for plotting note that the solution for y involves no radicals, and hence no values of x are to be excluded. Furthermore, the table should be extensive enough to show how the curve approaches its asymptotes.

To find both the horizontal and the vertical asymptotes it is not necessary to solve the equation for both x and y, as the horizontal asymptotes can be found from the solution for y by inspection. The reasoning is as follows: A horizontal asymptote occurs if there is a value of y for which x becomes infinite. Take the above solution for y in terms of x as an example:

$$y = \frac{2x^2 - x}{x^2 - 1}.$$

In this equation let x become infinite and see if y approaches any fixed value. For very large values of x the most important term in the numerator is $2x^2$, and the most important in the denominator is x^2. The value of the fraction, for very large values of x, will be approximately $2x^2$ divided by x^2, which is 2. As x becomes larger and larger the terms of lower degrees in numerator and denominator become relatively more and more insignificant, and the value of the fraction comes nearer and nearer to 2. Hence $y = 2$ is a horizontal asymptote.

PROBLEMS

In the following problems find the intercepts and the horizontal and vertical asymptotes and draw the curves:

1. $4xy + 2y - 2x + 1 = 0.$
2. $xy - x^2 + 4 = 0.$
3. $xy^2 - 4x + 4 = 0.$
4. $x^2y - 2y - x^2 = 0.$
5. $xy - 2x - y + 3 = 0.$
6. $xy^2 - y^2 - 4x + 1 = 0.$
7. $x^2y + x^2 - 4y - 1 = 0.$
8. $x^2y(y^2 - 4) = 1.$
9. $xy(x-1)(x-2) = 1.$
10. $y^3 + 3y^2 + x^2y - 3x^2 = 0.$

133. Direction of Bending. It has been seen (§ 71) that any function is increasing when its derivative is positive, and decreasing when its derivative is negative, and conversely. Let $y = f(x)$, and consider the derived function $y' = f'(x)$. As a consequence of the fact stated above, y' is increasing if *its* derivative, y'', is positive, and y' is decreasing if y'' is negative, and conversely. The geometric significance of this statement is important.

Let the figure represent the graph of $y = f(x)$. Then $y' = f'(x)$ will give the value of the slope of the graph at any point. At A the slope is positive and equal approximately to 3 (as estimated from the figure). Proceeding to the right along the curve (that is, as x increases), the graph shows that the slope decreases until the maximum

point B is reached, where $y' = 0$. From B to C the slope continues to decrease from zero to a value approximately -3. As x increases from a to c the slope y' decreases; hence the derivative of y', which is y'', is negative. From A to C the curve is bending downward, or is *concave downward.*

Proceeding from C to D along the curve, the slope increases from -3 (approximately) to zero, and from D to

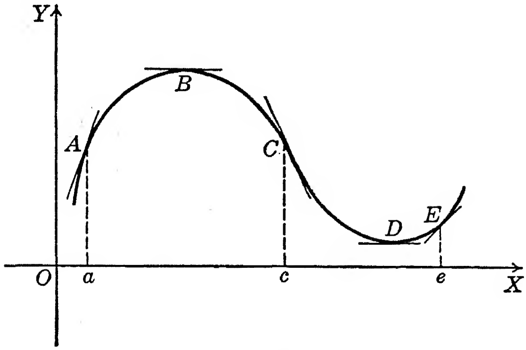


FIG. 120

E the slope continues to increase from zero to a value approximately 1. As x increases from c to e the slope y' increases; hence y'' is positive. From C to E the curve is bending upward, or is *concave upward.*

The preceding discussion justifies the following statement of the geometric significance of the second derivative.

If the graph of the function $y = f(x)$ is concave upward, the second derivative of y with respect to x is positive, and conversely; if the graph is concave downward, the second derivative is negative, and conversely.

134. Third Test for Maxima and Minima. The figure of the preceding section shows that the graph of a function is concave downward at a maximum point and concave

upward at a minimum point. Hence the second derivative is negative for a maximum value and positive for a minimum value. This furnishes a third test for distinguishing between maxima and minima, which may be used instead of the two given in § 74.

EXAMPLE. Given the equation $y = \frac{x^3}{3} - \frac{3x^2}{2} + 3$. Find the maximum and minimum points of its graph. Find also the values of x for which the graph is concave upward and those for which it is concave downward.

Solution. Finding the first and second derivatives, we have

$$y' = x^2 - 3x,$$
$$y'' = 2x - 3.$$

Setting $y' = 0$, we obtain $x = 0$ or 3, the critical values. When $x = 0$, $y'' = -3$; hence $x = 0$ gives a maximum point (0, 3). When $x = 3$, $y'' = +3$. Hence $x = 3$ gives a minimum point $(3, -\frac{3}{2})$.

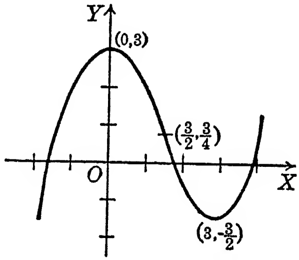


FIG. 121

The curve is concave downward when $y'' < 0$ or when $x < \frac{3}{2}$; it is concave upward when $y'' > 0$ or when $x > \frac{3}{2}$.

135. Points of Inflection. *A point of inflection is a point at which a curve changes its direction of bending.** That is,

* In connection with this definition it should be remembered that we are considering only functions which are single-valued and continuous, and which have continuous first and second derivatives. Consider the circle $x^2 + y^2 = a^2$. The points $A(a, 0)$ and $B(-a, 0)$ separate an arc of the curve which is concave upward from an arc which is concave downward. But A and B are not points of inflection. The function of x is $y = \pm\sqrt{a^2 - x^2}$, which is not single-valued, and $\frac{dy}{dx}$ becomes infinite at A and B.

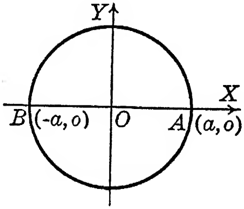


FIG. 122

as the tracing point moves from left to right the curve changes from concave downward to concave upward, or vice versa, at a point of inflection. In the figure of § 133, C is a point of inflection. In the preceding example the inflectional point is $(\frac{3}{2}, \frac{3}{4})$. Since the discussion of § 133 shows that y'' is negative at the left of C and positive at the right, y'' will in general* be equal to zero at C. Hence, *to find the points of inflection set the second derivative of y with respect to x equal to zero and solve for x.*

The tangent to the curve at a point of inflection is called an *inflectional tangent.* It should be observed that an inflectional tangent cuts through the curve at the point of tangency, and that it also approximates the curve very closely for a considerable distance on either side of the point of tangency. For this reason it is helpful in tracing a curve to draw the inflectional tangents with their proper slopes.

Exercise. Show that the slope is a maximum or a minimum at a point of inflection.

PROBLEMS

In the following problems (a) find the maximum, minimum, and inflectional points; (b) find the slope of each inflectional tangent and draw it; (c) draw the curve.

1. $y = \frac{x^3}{3} - x^2 + 2.$
2. $6y = 6 + 30x - 2x^3.$
3. $6y = x^3 - 12x - 6.$
4. $2y = 2x^3 + 3x^2.$
5. $6y = x^3 - 6x^2 + 12x.$
6. $y = x(x-1)(x-2).$
7. $y = \frac{x^4}{4} - 2x^2 + 2.$
8. $20y = x^4 - 24x^2 - 12.$
9. $6y = 6 + 18x^2 + 8x^3 - 3x^4.$
10. $y = (x^2 - 1)(x^2 - 4).$

* This statement is always true if y'' varies continuously. In certain curves, which will not be considered in this book, points of inflection exist at which y'' is not zero; that is, y'' changes sign by becoming infinite instead of zero.

11. $20y = x^5 - 10x^2.$

12. $15y = 3x^5 - 25x^3 + 60x.$

13. $y = x + \dfrac{4}{x}.$

14. $y = x^2 + \dfrac{2}{x}.$

15. $y = x + \dfrac{4}{x^2}.$

16. $y = x^3 + \dfrac{1}{x^2}.$

17. Show that the following curves have no point of inflection: (*a*) the ellipse, (*b*) the hyperbola, (*c*) the parabola.

18. Find the points of inflection of the curve whose equation is $y = \dfrac{x}{x^2+1}$ and show that they lie on a straight line.

19. Show that the curve whose equation is $x^3 + y^3 = 1$ has a point of inflection where it crosses the y-axis.

136. Information given by the Derivatives. Before passing on to the general discussion of an equation, we summarize for convenience the results of §§ 133–135 and Chapters II–IV, which bear on the tracing of a curve. These are as follows:

1. If y' is positive (negative), the curve is rising (falling).
2. If y'' is positive (negative), the curve is concave upward (downward).
3. If $y' = 0$ and y'' is positive (negative), the curve has a minimum (maximum) point.
4. If $y'' = 0$, the curve has a point of inflection.

The best way of making use of the information found in this way is given in the next section.

137. Discussion of an Equation. The discussion of an equation is resorted to for the purpose of getting all possible advance information about the properties of the curve before making a table of values and drawing the

curve. The five points to be considered are those which have been mentioned in §§ 128–135; namely,

1. Intercepts on the axes.
2. Symmetry.
3. Extent.
4. Horizontal and vertical asymptotes.
5. Maximum, minimum, and inflectional points.

It is not to be expected that complete information on all five points will be obtained from every equation. For example, to determine the extent of a curve, it is necessary to solve the equation algebraically for both x and y. This will not be done if the equation is of higher degree than the second. But partial information about the extent may be obtained if the equation is solved for one variable only. Again, the determination of intercepts may involve the solution of a numerical equation of higher degree than the second. This will not be undertaken unless the equation can be readily factored, because the results do not justify the labor.

But in every case of curve-tracing the five questions of the discussion should be answered as far as practicable. As each fact concerning the curve appears in the discussion it should be indicated, if possible, on the plotting-paper. Thus when the first question has been answered, the intercepts should be marked on the coördinate axes. If certain values are to be excluded, this fact should be indicated as in § 131. If the answer to the fourth question shows that the curve has horizontal or vertical asymptotes, these should be drawn on the plotting-paper to serve as guide lines in tracing the curve. The maximum, minimum, and inflectional points should be carefully plotted and a short tangent line with proper slope drawn through each. The knowledge

thus gained frequently makes it possible to sketch the curve completely. If the information obtained by the discussion is not sufficient for drawing the curve completely, point-by-point plotting must be used.

EXAMPLE 1. Discuss the equation $3y = x^3 + 3x^2 - 9x$ and trace its graph.

Solution. Solving for y, we obtain

$$y = \frac{x^3}{3} + x^2 - 3x. \tag{1}$$

We cannot solve for x, since the equation is a cubic in x.

Differentiating, we have

$$y' = x^2 + 2x - 3 = (x + 3)(x - 1), \tag{2}$$

$$y'' = 2x + 2. \tag{3}$$

We now take up the five points of the discussion in order:

Discussion

1. *Intercepts.* If $x = 0$, $y = 0$.

If $y = 0$, $x^3 + 3x^2 - 9x = 0$; then $x = 0$ or $x^2 + 3x - 9 = 0$, whence $x = -4.9$ or 1.9.

2. *Symmetry.* There is no symmetry with respect to the axes or the origin.

3. *Extent.* The solution for y in terms of x involves no radicals; hence no values of x are to be excluded. As $x \to +\infty$, $y \to +\infty$; hence the curve extends indefinitely to the right and upward. As $x \to -\infty$, $y \to -\infty$; hence the curve extends indefinitely to the left and downward.

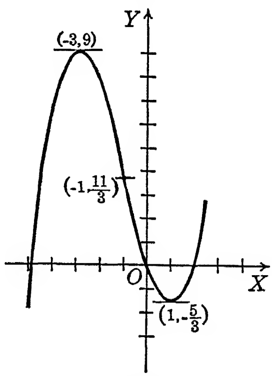


FIG. 123

4. *Asymptotes.* The solution for y does not involve fractions with x in the denominator; hence there are no vertical or horizontal asymptotes.

5. *Maximum, minimum, and inflectional points.* Setting $y' = 0$, we find $x = -3$ and 1. When $x = -3$, $y = 9$, and y'' is negative; hence $(-3, 9)$ is a maximum point. When $x = 1$, $y = -\frac{5}{3}$, and y'' is positive; hence $(1, -\frac{5}{3})$ is a minimum point. Setting $y'' = 0$, we find $x = -1$, $y = \frac{11}{3}$, and $y' = -4$. The curve has a point of inflection at $(-1, \frac{11}{3})$, and the slope of the inflectional tangent is -4.

The facts shown by the discussion are sufficient to trace the curve completely.

EXAMPLE 2. Discuss the equation $x^2y - x^2 + 4 = 0$ and trace its graph.

Solution. We are given

$$x^2y - x^2 + 4 = 0. \tag{1}$$

Solving for y,
$$y = 1 - \frac{4}{x^2}. \tag{2}$$

Solving for x,
$$x = \pm\sqrt{\frac{4}{1-y}}. \tag{3}$$

Differentiating (2),
$$y' = \frac{8}{x^3}, \tag{4}$$

$$y'' = -\frac{24}{x^4}. \tag{5}$$

Discussion

1. *Intercepts.* If $y = 0$, $x = \pm 2$. If $x = 0$, equation (1) becomes $4 = 0$, which is impossible. Hence the curve does not cross the y-axis.

2. *Symmetry.* The curve is symmetrical with respect to the y-axis. It is not symmetrical with respect to the x-axis or the origin.

3. *Extent.* Equation (2) shows that no values of x (except 0) need be excluded. Hence the curve extends indefinitely to the right and to the left. Equation (3) shows that values of y greater than 1 must be excluded. Hence there is no

part of the curve above the line $y=1$, but it extends indefinitely downward.

4. *Asymptotes.* Equation (2) shows that y becomes infinite when x approaches zero. Hence the y-axis is a vertical asymptote. Equation (3) shows that $y = 1$ is a horizontal asymptote.

5. *Maximum, minimum, and inflectional points.* Setting $y' = 0$ gives $8=0$, which is impossible. Hence there is no maximum or minimum point. Setting $y'' = 0$ gives $24 = 0$, which is impossible. Hence there is no inflectional point.

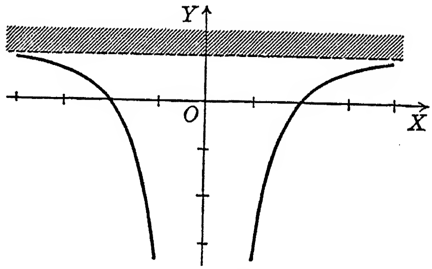


FIG. 124

Since there are no critical values, it is necessary to calculate a few values besides the intercepts in order to make an accurate drawing, although the general shape is obvious from the discussion.

EXAMPLE 3. Discuss the equation $x^2y + a^2y - 4\,a^2x = 0$ and trace its graph.

Solution. We are given

$$x^2y + a^2y - 4\,a^2x = 0. \tag{1}$$

Solving for y,
$$y = \frac{4\,a^2x}{x^2 + a^2}. \tag{2}$$

Solving for x,
$$x = \frac{2\,a^2 \pm a\sqrt{4\,a^2 - y^2}}{y}. \tag{3}$$

Differentiating (2),
$$y' = \frac{4\,a^2(a^2 - x^2)}{(x^2 + a^2)^2}, \tag{4}$$

$$y'' = \frac{8\,a^2x\,(x^2 - 3\,a^2)}{(x^2 + a^2)^3}. \tag{5}$$

Discussion

1. *Intercepts.* If $y = 0$, $x = 0$, and if $x = 0$, $y = 0$. Hence the curve cuts the coördinate axes only at the origin.

2. *Symmetry.* The curve is symmetrical with respect to the origin, but not with respect to either axis.

3. *Extent.* Equation (2) shows that no values of x need be excluded. Hence the curve extends indefinitely to the right and to the left. Equation (3) shows that y^2 cannot be greater than $4a^2$. Hence the curve lies entirely in the horizontal strip bounded by the lines $y = 2a$ and $y = -2a$.

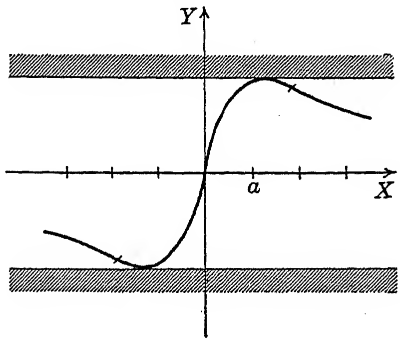


FIG. 125

4. *Asymptotes.* Equation (2) shows that there is no vertical asymptote. This is also apparent from the fact that y cannot exceed $2a$ in numerical value. Equation (3) shows that $y = 0$ is a horizontal asymptote.

5. *Maximum, minimum, and inflectional points.* Setting $y' = 0$ gives $x = +a$ and $-a$. For $x = +a$, $y = 2a$ and y'' is negative. Hence $(a, 2a)$ is a maximum point. For $x = -a$, $y = -2a$ and y'' is positive. Hence $(-a, -2a)$ is a minimum point.

Setting $y'' = 0$ gives $x = 0, +a\sqrt{3}, -a\sqrt{3}$. Hence the curve has three inflectional points: $(-a\sqrt{3}, -a\sqrt{3})$, with slope $-\frac{1}{2}$; $(0, 0)$, with slope 4; $(a\sqrt{3}, a\sqrt{3})$, with slope $-\frac{1}{2}$.

The curve is concave upward between $(-a\sqrt{3}, -a\sqrt{3})$ and $(0, 0)$ and to the right of $(a\sqrt{3}, a\sqrt{3})$. It is concave downward between $(0, 0)$ and $(a\sqrt{3}, a\sqrt{3})$ and to the left of $(-a\sqrt{3}, -a\sqrt{3})$.

Choosing any convenient numerical value for a, the curve can be sketched from the information given by the discussion.

PROBLEMS

Discuss the following equations and trace their graphs:

1. $x^3 + y - 1 = 0.$
2. $12y = 24 + 24x^2 - x^4.$
3. $xy + y - x = 0.$
4. $x^2y + y - x^2 = 0.$
5. $y = x^2(x^2 - 4).$
6. $y = x^2(4 - x^2).$
7. $y = x + \frac{4}{x}.$
8. $y = x + \frac{4}{x^2}.$
9. $8x^2y - x^5 + 32 = 0.$
10. $y = x^2 + \frac{4}{x}.$
11. $x^3y - 8y - x^3 = 0.$
12. $y = \frac{8a^3}{x^2 + 4a^2}.$
13. $y = \frac{2x^2}{x^2 + 4}.$
14. $y = x^5 - \frac{3x^3}{5}.$
15. $y = x^5 - x.$
16. $y = x^5 - x^2.$

138. More Complicated Curves. When the solution of the equation for y involves radicals, the derivatives are frequently so complicated that the labor of finding all maximum, minimum, and inflectional points is not justified for the purpose of sketching the curve. In these cases it is best to discuss the equation for intercepts, symmetry, extent, and asymptotes, and then to make a sketch from a table of values. When it is necessary to use the derivatives, care must be exercised in handling the double signs which occur when the solution for y involves an even root. It is best to operate with the positive sign alone. The results for the negative sign can usually be inferred from those obtained for the positive sign.

Example 1. Discuss the equation $y^2 = ax^3$ and trace its graph.

Solution. We are given

$$y^2 = ax^3. \tag{1}$$

Solving for y,

$$y = \pm\sqrt{ax^3}. \tag{2}$$

Solving for x, $$x=\sqrt[3]{\frac{y^2}{a}}. \tag{3}$$

Differentiating (2), taking the positive sign before the radical,

$$y'=\tfrac{3}{2}\sqrt{ax}, \tag{4}$$

$$y''=\frac{3a}{4\sqrt{ax}}. \tag{5}$$

Discussion

1. *Intercepts.* If $x=0$, $y=0$, and if $y=0$, $x=0$. Hence the curve cuts the axes only at the origin.

2. *Symmetry.* The curve is symmetrical with respect to the x-axis only.

3. *Extent.* Equation (2) shows that x must have the same sign as a. We will assume that a is positive; then all negative values of x must be excluded, and there is no part of the curve to the left of the y-axis. Equation (3) shows that no values of y need be excluded; hence the curve extends indefinitely to the right and upward and indefinitely to the right and downward.

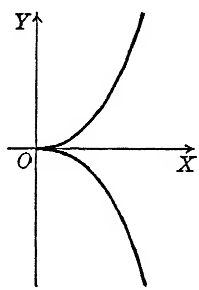


FIG. 126

4. *Asymptotes.* The curve has no vertical or horizontal asymptotes.

5. *Maximum, minimum, and inflectional points.* Setting $y'=0$, we find $x=0$. For $x=0$, $y=0$ and y'' becomes infinite. The curve is tangent to the x-axis at the origin. Since the curve does not extend to the left of the origin, the usual tests for maxima and minima cannot be made.

The second derivative is always positive; hence there is no inflectional point, and the branch of the curve above the x-axis is always concave upward.

When the negative sign is taken with the radical, the results may be inferred by symmetry. The branch of the curve below the x-axis is always concave downward and is tangent to the x-axis at the origin.

The two branches of this curve are tangent to the same line at the same point, forming a "sharp" point on the curve. Such a point is called a *cusp*.

EXAMPLE 2. Discuss the equation $(a-x)y^2-(a+x)x^2=0$ and trace its graph.

Solution. We are given

$$(a-x)y^2-(a+x)x^2=0. \tag{1}$$

Solving for y,
$$y=\pm x\sqrt{\frac{a+x}{a-x}}. \tag{2}$$

The solution for x is not practicable.

Discussion

We will assume that a is positive.

1. *Intercepts.* If $x=0$, $y=0$. If $y=0$, $x=0$ or $-a$.

2. *Symmetry.* The curve is symmetrical with respect to the x-axis only.

3. *Extent.* Equation (2) shows that $x>a$, and $x<-a$ must be excluded. Hence the curve lies entirely within the vertical strip bounded by the lines $x=a$ and $x=-a$.

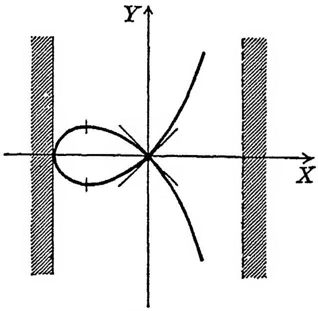


FIG. 127

4. *Asymptotes.* The line $x=a$ is a vertical asymptote. There is no horizontal asymptote, because x cannot exceed a numerically.

Taking the positive sign with the radical, equation (2) shows that y has the same sign as x. Hence the curve lies below the x-axis between $-a$ and 0 and above the x-axis between 0 and a, approaching the positive end of the asymptote $x=a$. Obviously the curve must have a minimum point between $-a$ and 0. The other half of the curve, obtained by taking the minus sign in equation (2), can be drawn in by symmetry.

When a numerical value of a is given, the shape of the curve is readily obtained by plotting a few points. If it is desired to locate accurately the minimum point between $-a$ and 0, we find, from equation (2),

$$y' = \frac{a^2 + ax - x^2}{(a - x)\sqrt{a^2 - x^2}}.$$

Setting $y' = 0$ gives

$$x = \frac{a}{2}(1 \pm \sqrt{5}).$$

The positive sign must be rejected, since x cannot be greater than a. Hence the minimum point is

$$x = \frac{a}{2}(1 - \sqrt{5}) = -0.618\,a, \qquad y = -0.300\,a.$$

By symmetry there is a maximum point $(-0.618\,a, +0.300\,a)$.

Example 3. Discuss the equation $x^2 - xy + y^2 = 12$ and trace its graph.

Solution. We are given

$$x^2 - xy + y^2 = 12. \tag{1}$$

Solving for y, $$y = \frac{x \pm \sqrt{48 - 3x^2}}{2}. \tag{2}$$

Solving for x, $$x = \frac{y \pm \sqrt{48 - 3y^2}}{2}. \tag{3}$$

Discussion

1. *Intercepts.* If $x = 0$, $y = \pm\sqrt{12}$. If $y = 0$, $x = \pm\sqrt{12}$.

2. *Symmetry.* The curve is symmetrical with respect to the origin, but not with respect to either axis.

3. *Extent.* Equation (2) shows that x cannot be greater numerically than 4. Equation (3) shows that y cannot be greater numerically than 4. Hence the curve lies entirely within the square bounded by the lines $x = 4$, $x = -4$, $y = 4$, $y = -4$.

4. *Asymptotes.* There can be no asymptotes, because neither x nor y can become infinite.

Taking the positive sign with the radical in equation (2), the table of values is computed for half of the curve.

x	y
-4	-2
-3	0.79
-2	2
-1	2.85
0	3.46
1	3.85
2	4
3	3.79
4	2

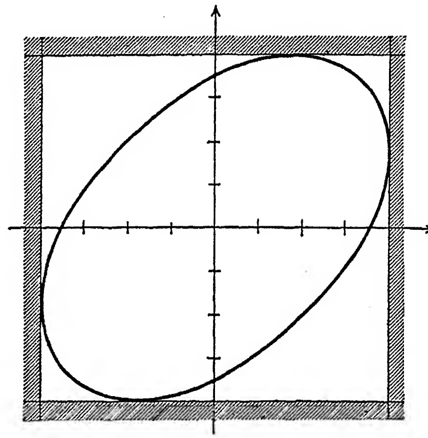


FIG. 128

The other half of the curve can be drawn by symmetry. It is obvious from the figure that $(2, 4)$ is a maximum point and $(-2, -4)$ is a minimum point.

PROBLEMS

Discuss the following equations and trace their graphs:

1. $x^3 + y^2 - 1 = 0.$
2. $y^2 + x^3 - x = 0.$
3. $xy^2 + y^2 - x = 0.$
4. $y^2 = x^2(x^2 - 16).$
5. $x^2y^2 + y^2 - x^2 = 0.$
6. $y^2 = x^2(16 - x^2).$
7. $x^2 - 2xy + y^2 - x - y = 0.$
8. $x^2 + xy + y^2 = 4.$
9. $x^2 + 3xy + y^2 = 4.$
10. $y^2(2a - x) = x^3.$
11. $\sqrt{x} + \sqrt{y} = \sqrt{a}.$
12. $(y - x^2)^2 = x^5.$
13. $y^2(a - x) = x^2(a + x).$
14. $y^2(x^2 - 4x + 3) = x.$
15. $y^2 = (4 - x)(3 - x)(2 - x).$
16. $x^4 + y^2 = 16.$

139. Circle of Curvature. In a previous chapter the direction of a curve at any point was defined as that of the tangent at that point. It has also been noted that an inflectional tangent fits a curve very closely at the point of contact. Tangents at other points, however, do not fit the curve closely, especially if the direction of the curve is changing rapidly. A better approximation can be obtained by means of a properly drawn circle.

The conditions determining such a circle are fairly obvious. First, it must be tangent to the curve; hence its center will be on the normal drawn at the given point. Moreover, for the approximation to be close, the slope of the circle must change at the same rate as that of the curve; otherwise the circle will at once diverge rapidly from the curve, just as the tangent line does. We therefore have the following definition:

The circle of curvature at any point of a curve is the circle which is tangent to the curve at that point and whose slope changes at the same rate as that of the curve. In brief, the requirement is that, for the point of tangency, y' and y'' shall be the same for both curve and circle.

140. Radius of Curvature. The radius of the circle of curvature at a point is called the *radius of curvature* of the curve at that point. To obtain a formula for this we proceed as follows:

Let $P(x, y)$ be the point on the curve $y=f(x)$, $C(h, k)$ the center of the circle of curvature, and R its radius.

Since the circle is tangent to the curve at P, the coördinates of this point satisfy the equation of the circle

$$(x-h)^2+(y-k)^2=R^2. \tag{1}$$

Differentiating, $$\frac{dy}{dx}=-\frac{x-h}{y-k}.$$

But the circle is *tangent* to the curve at P; hence, if y' denotes the derivative of $y=f(x)$,

$$y' = -\frac{x-h}{y-k}. \tag{2}$$

Moreover, the definition requires that the slopes of the circle and of the curve change at the same rate. Hence their second derivatives are the same; that is,

$$y'' = -\frac{(y-k)-(x-h)y'}{(y-k)^2}$$

$$= -\frac{(y-k)+\dfrac{(x-h)^2}{y-k}}{(y-k)^2}.$$

After simplification this becomes

$$y'' = -\frac{(y-k)^2+(x-h)^2}{(y-k)^3}. \tag{3}$$

To obtain a formula for R we must eliminate h and k from equations (1), (2), and (3). We first eliminate h. Using equation (1) in (3), we obtain

$$y'' = -\frac{R^2}{(y-k)^3}. \tag{4}$$

Squaring (2) and adding 1 to both sides, we get

$$1+y'^2 = 1+\frac{(x-h)^2}{(y-k)^2} = \frac{(y-k)^2+(x-h)^2}{(y-k)^2}.$$

Using equation (1) again, this becomes

$$1+y'^2 = \frac{R^2}{(y-k)^2}. \tag{5}$$

To eliminate k we square both sides of equation (4), cube both sides of equation (5), and divide the second result by the first. This gives

$$R^2 = \frac{(1+y'^2)^3}{y''^2};$$

whence
$$R = \pm\frac{(1+y'^2)^{\frac{3}{2}}}{y''},$$

which is the formula for R.

By substituting this in the earlier equations, formulas for h and k may be obtained, but it is usually better to get the center by simply measuring off R along the normal from P. The center is above or below the curve, according as y'' is positive or negative at P; but it is better simply to use the fact that C must be on the concave side of the curve.

EXAMPLE 1. Draw the circle of curvature of $y = x^3$ at the point (1, 1).

Solution. Differentiating,

$$y' = 3x^2\Big]_{x=1} = 3;$$

$$y'' = 6x\Big]_{x=1} = 6.$$

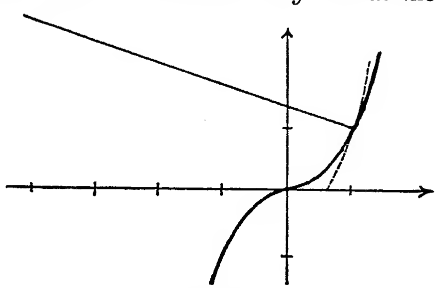


FIG. 129

Substituting in the formula,

$$R = \frac{(1+9)^{\frac{3}{2}}}{6} = \frac{10\sqrt{10}}{6} = 5.3.$$

The slope of the normal is $-\frac{1}{3}$, and the curve is concave upward at this point. In the figure note that the circle and the curve are indistinguishable near the point of contact; also that the circle crosses the curve at the point of contact.

EXAMPLE 2. Find the radius of curvature of the parabola $y^2 = 2px$ at the vertex.

Solution. Differentiating,

$$2yy' = 2p, \quad \text{or} \quad y' = \frac{p}{y};$$

$$y'' = -\frac{p}{y^2}\frac{dy}{dx} = -\frac{p^2}{y^3}.$$

At the vertex, $y = 0$ and both y' and y'' are infinite. In such cases substitute the general expressions for y' and y'' in the formula for R, and simplify before putting in the values of x and y. Doing this, we have

$$R = \frac{\left(1 + \frac{p^2}{y^2}\right)^{\frac{3}{2}}}{\frac{p^2}{y^3}}$$

$$= \frac{(y^2 + p^2)^{\frac{3}{2}}}{y^3} \frac{y^3}{p^2},$$

or $$R = \frac{(y^2 + p^2)^{\frac{3}{2}}}{p^2}.$$

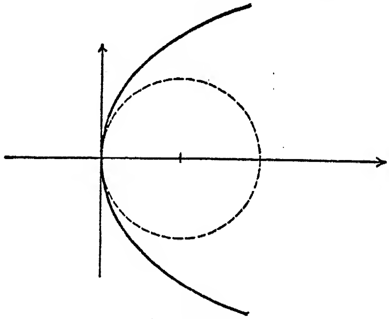


FIG. 130

If we *now* set $y = 0$, we get the desired radius, which is $R = p$.

PROBLEMS

1. Draw the following curves with their circles of curvature at the points indicated:

a. $4y = x^2$, $(2, 1)$.

b. $y = x^3 - 6x^2 + 9x$, $(3, 0)$.

c. $xy = -24$, $(4, -6)$.

d. $y = x^2 - 4x + 2$, $(1, -1)$.

e. $x^2 - y^2 = 12$, $(4, 2)$.

f. $y^2 = x^3$, $(1, 1)$.

g. $x^2 + 4y^2 = 8$, $(2, -1)$.

h. $y = x^2(4 - x^2)$, $(0, 0)$.

i. $y = x^2 - x^3$, $(0, 0)$, $(1, 0)$.

j. $x^2 = 2py$, $\left(p, \frac{p}{2}\right)$.

2. Find the radius of curvature for each of the following curves at the point (x, y):

a. $b^2x^2 + a^2y^2 = a^2b^2$. *c.* $y = ax^3$.
b. $b^2x^2 - a^2y^2 = a^2b^2$. *d.* $y^2 = ax^3$.

3. Show that the radius of curvature of a straight line is infinite.

4. Show that at a point of inflection the radius of curvature of any curve is infinite.

5. Using the result of Problem 2, *a*, show that the radii of curvature of an ellipse at the ends of its axes are $\frac{b^2}{a}$ and $\frac{a^2}{b}$. Draw an ellipse with these four circles of curvature.

6. Find the radius of curvature of the parabola $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$.

7. Find the radius of curvature of the witch $y = \dfrac{8a^3}{x^2 + 4a^2}$ at its maximum point.

8. For each of the following curves find the point where the radius of curvature is a maximum or a minimum, and illustrate by a figure:

a. $y = x^3$. *c.* $xy = 24$.
b. $y = 4x - x^2$. *d.* $y^2 = x^3$.

9. For each of the following equations find the radius of curvature for the given value of the parameter, and draw the figure.

HINT. In finding y'' remember that $y'' = \dfrac{dy'}{dx} = \dfrac{dy'}{dt}$ divided by $\dfrac{dx}{dt}$ (see §§ 56, 59, 89).

a. $x = t^2$, $2y = t$, $t = 1$.
b. $x = t^2$, $y = t^3$, $t = 1$.
c. $x = t$, $y = \dfrac{24}{t}$, $t = 3$.

141. Applications. 1. *Small Arcs of Curves.* The length of a small arc of a curve may sometimes be approximated by means of its circle of curvature. As an example, let us find the length of an arc of 1° measured along a meridian

at the equator. Since a meridian is an ellipse with its vertices at the equator, we know that its radius of curvature at this point is $\frac{b^2}{a}$ (see Problem 5 above). The equatorial radius of the earth is 3963 mi. and the polar radius is 3950 mi.; hence $a = 3963$ and $b = 3950$. Therefore $R = \frac{b^2}{a} = 3937$. Now an arc of 1° of the elliptic meridian is relatively short, and may therefore be closely approximated by finding a similar arc of the circle of curvature. This is easily seen to be

$$\frac{2\pi R}{360} = \frac{(3.1416)(3937)}{180} = 68.72 \text{ mi.}$$

Exercise. Show that an arc of 1° at one of the poles is about 69.39 mi.

2. *Railroad Curves.* It is well known that railroad tracks are "banked" at curves (that is, the outer rail is higher than the inner one) so as to avoid undue pressure on the outer rail and to prevent swiftly moving trains from jumping the track. Moreover, the banking is steeper when the radius of curvature is smaller. The same thing is true of properly constructed automobile roads.

If a straight track were joined immediately to a circular track there would be a sudden change in direction, with the consequent banking of the track, which is undesirable. For this reason engineers have sought a transition, or easement, curve with which to connect straight and circular tracks. The requirements which this curve must fulfill are obvious. It must have at one end the same radius of curvature as the straight line, and at the other

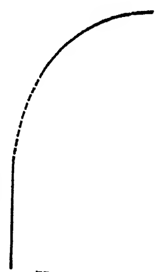


Fig. 131

end the same radius of curvature as the circle. The radius of curvature of the straight line is infinite, and the radius of the circle will be some finite quantity. Hence a curve is required whose radius of curvature will be infinite at one point and will decrease, as we move along the curve, to the desired radius of the circle. The simplest curve possessing these properties is the cubical parabola $y = ax^3$, whose radius of curvature at any point is

$$R = \frac{(1 + 9\,a^2x^4)^{\frac{3}{2}}}{6\,ax}.$$

At $x = 0$, R is infinite; by properly choosing a and x, R can be made equal to the desired radius of the circle. Hence the cubical parabola is used as the theoretical transition curve. In actual practice the difficulties of laying out this curve exactly are avoided by the use of various approximate methods suited to the degree of accuracy required.

CHAPTER IX

INDEFINITE INTEGRALS

142. Integration. The process of differentiation enables us to find the differential (or derivative) of a given function. Integration is the inverse process of finding the function when the differential (or derivative) is given.

For example, if the given function is x^3, the differential is found to be $3\,x^2dx$. Inversely, if the differential $3\,x^2dx$ is given, it is clear that a function having this differential is x^3. But this is not the only function having the given differential. Obviously $3\,x^2dx$ is the differential of x^3-7, of x^3+10, and, in general, of x^3+C, where C is any constant.

Notation. In the example above, x^3+C is called the *integral* of $3\,x^2dx$, and this is indicated by the notation

$$\int 3\,x^2dx = x^3 + C.$$

The function $3\,x^2$ is called the *integrand.*

In general the notation

$$\int f(x)\,dx = F(x) + C$$

means that $F(x)+C$ is the integral of $f(x)\,dx$; $f(x)$ is the integrand and, by definition,

$$dF = f(x)\,dx, \quad \text{or} \quad \frac{dF}{dx} = f(x).$$

In words, the integral may be defined as follows: *The indefinite integral of a given differential expression is the*

*function whose differential is the given expression.** The integral is called *indefinite* because the constant C can have any value. In order to determine a value for C, additional data must be given, as explained in the next chapter.

143. Formulas for Integration. Since integration is an inverse process, the formulas are obtained by inverting the formulas for differentiation. In fact, every solution of a problem in differentiation yields a formula for integration. For practical use, tables of integrals have been compiled containing many forms. For present purposes we give only four formulas for integration:

$$\text{(I)} \qquad \int du = u + C.$$

$$\text{(II)} \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

$$\text{(III)} \qquad \int a\,du = a\int du.$$

$$\text{(IV)} \qquad \int u^n du = \frac{u^{n+1}}{n+1} + C, \text{ if } n \neq -1.$$

In these formulas u denotes a function of x (or some other independent variable).

Formula I merely states the definition of an integral.

Formula II states that the integral of a sum of differentials is equal to the sum of the integrals of the differentials.

In Formula III, a is any constant, and this formula shows that a constant can be moved from one side of the integral sign to the other. The student is warned that a *variable* cannot be moved from one side of the integral sign to the other.

* This definition implies that a given differential can have only one integral. That this is true is proved in more advanced courses, where it is shown that two functions which have the same differential differ only by a constant.

Formula IV is the *power formula*, in which n is any constant except -1. If the exponent of u in the integral is -1, the integral involves a logarithm, which will be considered later (Chapter XIV).

Proofs of the Formulas. It follows from the definition of an integral that each formula may be proved by showing that the differential of the right-hand side is the expression under the integral sign on the left. We give the proof for Formula II; the others are proved in a similar manner.

$$d\left[\int f(x)\,dx + \int g(x)\,dx\right] = d\int f(x)\,dx + d\int g(x)\,dx$$

$$= f(x)\,dx + g(x)\,dx \quad \text{by definition of integral}$$

$$= [f(x) + g(x)]\,dx.$$

144. Use of the Formulas. Any function consisting of algebraic expressions can be differentiated by direct application of the formulas already given. But comparatively few expressions can be integrated, and, for the present, only such expressions as can be reduced, by a proper transformation, to the exact form of one of the formulas of the preceding paragraph. Since the integrand is the derivative of an integral, the correctness of the work in any problem of integration can be tested by showing that the derivative of the result is equal to the integrand. The method of using the formulas will be illustrated by some examples:

EXAMPLE 1. Find $\int 5\,x^2 dx$.

Solution. $$\int 5\,x^2 dx = 5\int x^2 dx, \quad \text{by Formula III}$$

$$= \frac{5\,x^3}{3} + C. \quad \text{By Formula IV}$$

EXAMPLE 2. Find $\int (x^3 + 2x^2 + 3)\,dx$.

Solution. $\int (x^3 + 2x^2 + 3)\,dx$

$$= \int x^3 dx + \int 2x^2 dx + \int 3\,dx, \quad \text{by Formula II}$$

$$= \int x^3 dx + 2\int x^2 dx + 3\int dx, \quad \text{by Formula III}$$

$$= \frac{x^4}{4} + \frac{2x^3}{3} + 3x + C. \quad \text{By Formulas IV and I}$$

It should be noticed that a separate constant might be added with each of the three integrals of the second step of the solution. But this is unnecessary, since one arbitrary constant is equivalent to the sum of any number of constants.

EXAMPLE 3. Find $\int \frac{x^2 + 2}{\sqrt{x}}\,dx$.

Solution. Since the denominator is a monomial, we can divide term by term and apply the power formula to each term of the result. This gives

$$\int \frac{x^2 + 2}{\sqrt{x}}\,dx = \int (x^{\frac{3}{2}} + 2x^{-\frac{1}{2}})\,dx$$

$$= \int x^{\frac{3}{2}} dx + 2\int x^{-\frac{1}{2}}\,dx$$

$$= \tfrac{2}{5}x^{\frac{5}{2}} + 4x^{\frac{1}{2}} + C.$$

EXAMPLE 4. Find $\int \sqrt{x+1}\,dx$.

Solution. This integral can be found by the power formula. Comparison with Formula IV shows that $u = x + 1$, $du = dx$, and $n = \frac{1}{2}$. Hence

$$\int \sqrt{x+1}\,dx = \frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}(x+1)^{\frac{3}{2}} + C.$$

EXAMPLE 5. Find $\int \sqrt{8x}\,dx$.

Solution. This can be solved in two ways. The first is to factor out the constant, which gives

$$\int \sqrt{8x}\,dx = \sqrt{8}\int x^{\frac{1}{2}}dx = \sqrt{8}\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2x}{3}\sqrt{8x} + C.$$

A second solution is as follows: Let $u = 8x$; then $du = 8\,dx$. Since, by Formula III, a constant factor may be moved from one side of the integral sign to the other, we can multiply the integrand by 8 without changing the value of the given expression, provided that we place the compensating factor $\frac{1}{8}$ before the integral sign. We then have

$$\int \sqrt{8x}\,dx = \tfrac{1}{8}\int \sqrt{8x}\;8\,dx = \tfrac{1}{8}\int (8x)^{\frac{1}{2}}8\,dx.$$

The integral is now in the form of the power formula with $n = \frac{1}{2}$. Hence

$$\int \sqrt{8x}\,dx = \frac{1}{8}\cdot\frac{(8x)^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{(8x)^{\frac{3}{2}}}{12} + C.$$

This is easily seen to be equal to the result obtained above. (Compare Example 1, p. 110.)

EXAMPLE 6. Find $\int \frac{x\,dx}{\sqrt{a^2 - x^2}}$.

Solution. Let $u = a^2 - x^2$; then $du = -2x\,dx$.

Since, by Formula III, a constant factor may be moved from one side of the integral sign to the other, we may multiply numerator and denominator by -2 and write

$$\int \frac{x\,dx}{\sqrt{a^2 - x^2}} = -\frac{1}{2}\int \frac{-2x\,dx}{\sqrt{a^2 - x^2}} = -\frac{1}{2}\int (a^2 - x^2)^{-\frac{1}{2}}(-2x\,dx).$$

The integral is now in the form of the power formula with $n = -\frac{1}{2}$. Hence

$$\int \frac{x\,dx}{\sqrt{a^2 - x^2}} = -\frac{1}{2}\left[\frac{(a^2 - x^2)^{\frac{1}{2}}}{\frac{1}{2}}\right] + C = -\sqrt{a^2 - x^2} + C.$$

EXAMPLE 7. Find $\int \frac{dx}{(2x-3)^2}$.

Solution. Let $u = 2x - 3$; then $du = 2\,dx$.
Multiplying numerator and denominator by 2, we may write

$$\int \frac{dx}{(2x-3)^2} = \frac{1}{2}\int \frac{2\,dx}{(2x-3)^2} = \frac{1}{2}\int (2x-3)^{-2}(2\,dx).$$

The integral is now in the form of the power formula with $n = -2$. Hence

$$\int \frac{dx}{(2x-3)^2} = \frac{1}{2}\left[\frac{(2x-3)^{-1}}{-1}\right] + C = \frac{-1}{2(2x-3)} + C.$$

PROBLEMS

Integrate the following:

1. $\int 6\,dx$.
2. $\int \sqrt[3]{t^2}\,dt$.
3. $\int \frac{dx}{x^3}$.
4. $\int x^{-n}dx$.
5. $\int \frac{dy}{\sqrt{y}}$.
6. $\int (2-x)\,dx$.
7. $\int (x^4 - 3x^2 + x - 2)\,dx$.
8. $\int (a^2 - t^2)\,dt$.
9. $\int \left(x^2 - 3\sqrt{x} + \frac{6}{x^2}\right)dx$.
10. $\int (a + bx + cx^2)\,dx$.
11. $\int (t^2 - t^{-2})\,dt$.
12. $\int (1-x)(2+x^2)\,dx$.
13. $\int x^2(2-3x)\,dx$.
14. $\int \left(\frac{4}{x^3} + \frac{9}{x^4}\right)dx$.
15. $\int (2-3x)^2 x\,dx$.
16. $\int (2-x)\sqrt[3]{x}\,dx$.
17. $\int \frac{4-x^2}{x^4}\,dx$.
18. $\int (\sqrt{a} - \sqrt{x})^2 dx$.
19. $\int \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3 dx$.
20. $\int \left(\frac{3}{\sqrt{x}} - \frac{2}{\sqrt[3]{x}} - \frac{1}{x^2}\right)dx$.

21. $\int \sqrt{1-x}\,dx.$

22. $\int \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^2 \frac{dx}{x^{\frac{1}{3}}}.$

23. $\int x(a+bx^2)^{\frac{5}{4}}dx.$

24. $\int \left(\frac{y}{8} - 3\right)^2 dy.$

25. $\int \sqrt[3]{(2\,ax + x^2)}(a+x)\,dx.$

26. $\int \frac{(x^2-2)^3 dx}{x^2}.$

27. $\int \frac{6\,s\,ds}{\sqrt[3]{1-2\,s^2}}.$

28. $\int \sqrt{5\,x}\,dx.$

29. $\int \frac{dx}{\sqrt{3\,x}}.$

30. $\int \sqrt{x-2}\,dx.$

31. $\int \frac{dx}{\sqrt{5-x}}.$

32. $\int (3\,x-2)^2 dx.$

33. $\int (5-x)^3 dx.$

34. $\int (4-3\,x)^2 dx.$

35. $\int x(1+\sqrt{x})\,dx.$

36. $\int \left(\frac{x^2}{2} + \frac{2}{x^2}\right) dx.$

37. $\int \left(7\sqrt{x} + \frac{7}{\sqrt{x}}\right) dx.$

38. $\int \frac{(1+\sqrt{x})}{x^2}\,dx.$

39. $\int \frac{5\,x\,dx}{(x^2-3)^2}.$

40. $\int x\sqrt{3\,x}\,dx.$

41. $\int x\sqrt[3]{2\,x}\,dx.$

42. $\int \frac{3\,x\,dx}{\sqrt{5\,x}}.$

43. $\int x\sqrt{a^2-x^2}\,dx.$

44. $\int s\sqrt{2\,s^2-3}\,ds.$

45. $\int \frac{7\,t\,dt}{\sqrt{6-t^2}}.$

46. $\int 2\,y^3\sqrt{y^4+a^4}\,dy.$

47. $\int \frac{3\,v\,dv}{(9-4\,v^2)^3}.$

48. $\int z^2\sqrt{9+z^3}\,dz.$

145. Integration by Substitution. Many integrals can be reduced to the standard formulas by making a substitution which introduces a new independent variable. The following substitution of this type is frequently useful: when the integrand contains a radical of the form $\sqrt{a+bx}$, set $\sqrt{a+bx}=v$; then $a+bx=v^2$ and $b\,dx=2\,v\,dv$.

EXAMPLE. Integrate $x\sqrt{1+3\,x}\,dx$.

Solution. Let $\sqrt{1+3\,x}=v$; then $3\,dx=2\,v\,dv$, and $x=\dfrac{v^2-1}{3}$.

Now
$$\int x\sqrt{1+3\,x}\,dx=\int\left(\frac{v^2-1}{3}\right)(v)\frac{(2\,v\,dv)}{3}$$
$$=\frac{2}{9}\int(v^4-v^2)dv$$
$$=\frac{2}{9}\left[\frac{v^5}{5}-\frac{v^3}{3}\right]+C.$$

Hence, substituting the value of v,
$$\int x\sqrt{1+3\,x}\,dx=\frac{2}{9}\left[\frac{(1+3\,x)^{\frac{5}{2}}}{5}-\frac{(1+3\,x)^{\frac{3}{2}}}{3}\right]+C.$$

By factoring we can reduce the result to the form
$$\tfrac{2}{135}(9\,x-2)(1+3\,x)^{\frac{3}{2}}+C.$$
This reduction is sometimes, but not usually, desirable.

The method of substitution should not be used if the integration can be performed by the power formula.

PROBLEMS

Integrate the following:

1. $\displaystyle\int x\sqrt{a-x}\,dx.$

2. $\displaystyle\int\frac{x\,dx}{\sqrt{1+x}}.$

3. $\displaystyle\int\frac{x\,dx}{\sqrt{1+x^2}}.$

4. $\displaystyle\int(5-x^2)^2dx.$

5. $\displaystyle\int x^2\sqrt{1+x}\,dx.$

6. $\displaystyle\int x\sqrt{1+x^2}\,dx.$

7. $\int (1 + x^2)^3 dx.$

8. $\int \frac{x^2\,dx}{\sqrt{1-x}}.$

9. $\int \frac{x^2\,dx}{\sqrt{1-x^3}}.$

10. $\int \frac{t\,dt}{\sqrt{a+bt}}.$

11. $\int y\sqrt{a+by}\,dy.$

12. $\int y\sqrt{a+by^2}\,dy.$

13. $\int \frac{(x-2)^2 dx}{\sqrt{x}}.$

14. $\int \sqrt{2\,px}\,dx.$

15. $\int \sqrt{a+bx}\,dx.$

16. $\int t^3\sqrt{2-t}\,dt.$

17. $\int x^2\sqrt{4\,x+3}\,dx.$

18. $\int \sqrt[3]{8-x}\,dx.$

CHAPTER X

CONSTANT OF INTEGRATION

146. Initial Conditions. It has appeared in the preceding chapter that the process of integration introduces an arbitrary constant, and, consequently, a function is not completely determined when its differential is known. For the complete determination of a function by integration it is necessary to know the numerical value of the function corresponding to some numerical value of the independent variable. The particular values of the independent variable and of the function which serve to determine the constant of integration are known as *initial conditions.* The use of initial conditions to determine the constant of integration will be illustrated by an example.

EXAMPLE. Find the function $y = f(x)$ whose differential is $x^2 dx$ and which is equal to 5 when x is equal to 3.

Solution. Given $\qquad dy = x^2 dx.$

By integration
$$y = \int x^2 dx,$$
or
$$y = \frac{x^3}{3} + C.$$

Since $y = 5$ when $x = 3$, we have, by substitution,
$$5 = \tfrac{27}{3} + C;$$
whence
$$C = -4.$$

Substituting this value of C above, the final result is
$$y = \frac{x^3}{3} - 4.$$

PROBLEMS

In the following problems find the function whose differential is given and which satisfies the given initial conditions:

1. $dy = \frac{dx}{x^2}$, $y = 0$ when $x = 1$.

2. $dy = \sqrt{x}\,dx$, $y = 2$ when $x = 4$.

3. $dy = x\sqrt{x^2 + 16}\,dx$, $y = 5$ when $x = 3$.

4. $dy = \frac{dx}{\sqrt{5 - 4x}}$, $y = 4$ when $x = 1$.

5. $dy = \left(x^{\frac{1}{3}} + \frac{1}{x^{\frac{1}{3}}}\right)dx$, $y = 0$ when $x = 8$.

6. $ds = \sqrt{7\,t}\,dt$, $s = 0$ when $t = 0$.

7. $ds = (32\,t - 10)\,dt$, $s = 20$ when $t = 0$.

8. $ds = (t^2 - 2\sqrt{t})\,dt$, $s = 7$ when $t = 0$.

9. $dy = x\sqrt{9 + x}\,dx$, $y = 0$ when $x = 0$.

10. $dy = \frac{x\,dx}{\sqrt{5 + x}}$, $y = 1$ when $x = 4$.

11. $dy = \frac{x^2\,dx}{\sqrt{1 + x^3}}$, $y = 0$ when $x = 0$.

12. $dy = x\sqrt{25 - x^2}\,dx$, $y = 0$ when $x = 0$.

13. $ds = \frac{dx}{\sqrt{5 - x}}$, $s = -2$ when $x = -4$.

14. $dy = \sqrt{a^2 - x^2}\,x\,dx$, $y = 0$ when $x = a$.

15. $dy = \frac{x\,dx}{\sqrt{1 + x}}$, $y = \frac{10}{3}$ when $x = 3$.

16. $dz = (2 - x)\sqrt[3]{x}\,dx$, $z = -1$ when $x = 8$.

17. $dy = \frac{6 - x^2}{x^4}\,dx$, $y = 2$ when $x = 1$. Find the value of y when $x = -1$.

18. $ds = \sqrt{1-t}\,dt$, $s = \frac{1}{3}$ when $t = 0$. Find the value of s when $t = -1$.

19. $dA = \sqrt{2px}\,dx$, $A = \frac{p^2}{3}$ when $x = \frac{p}{2}$. Find the value of A when $x = 2p$.

20. $dz = (2 - y^2)^3 dy$, $z = -6$ when $y = 0$. Find the value of z when $y = 2$.

21. $dy = \sqrt{x+2}\,dx$, $y = \frac{1}{3}$ when $x = 2$. Find the value of y when $x = 7$.

22. $dz = \dfrac{dx}{\sqrt{x}}$, $z = -4$ when $x = 0$. Find the value of z when $x = 4$.

23. $dy = x\sqrt{9 + 4x^2}\,dx$, $y = 0$ when $x = 0$. Find the value of y when $x = 2$.

24. $dy = x\sqrt{2x+1}\,dx$, $y = 0$ when $x = 0$. Find the value of y when $x = 4$.

147. Curves with Given Slope. Suppose that x and y represent rectangular coördinates and that the derivative of y with respect to x is known in terms of x and y. Thus, let

$$\frac{dy}{dx} = f(x, y).$$

This equation is called a *differential equation*, because it involves a derivative, and it gives the slope of a curve at every point for which $f(x, y)$ has a real value. In order to determine the equation of a curve having the given slope at every point it is obvious that an integration must be performed. Since the process of integration introduces an arbitrary constant, the differential equation represents a *system of curves*. A particular curve of the system can be determined by requiring that it shall pass through a given point. The method of finding the equations of curves having given slopes will be illustrated by two examples.

EXAMPLE 1. Find the equation of the curve whose slope at every point is equal to its abscissa and which passes through the point $(0, -4)$.

Solution. By the conditions of the problem,

$$\frac{dy}{dx} = x, \qquad (1)$$

whence $$dy = x\,dx$$

and $$y = \int x\,dx.$$

Hence $$y = \frac{x^2}{2} + C. \qquad (2)$$

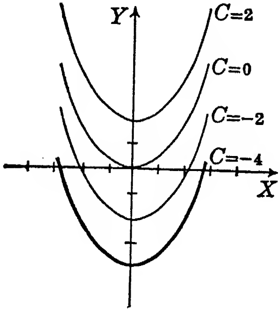


FIG. 132

The value of C is determined by substituting $x = 0$, $y = -4$, in equation (2):

$$-4 = 0 + C.$$

Hence $C = -4$, and, substituting this value of C in equation (2), the final result is $y = \frac{x^2}{2} - 4$.

Equation (2), which is the equivalent of the differential equation (1), represents a system of parabolas, as shown in the figure. The heavy curve of the system is the one which satisfies the given initial conditions.

EXAMPLE 2. Find the equation of the curve which passes through $(3, 2)$ and whose slope at every point is given by

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Solution. In order to integrate the given equation it is necessary first to *separate the variables*; that is, to write the given equation in a form such that each term contains only x or only y. Thus, clearing of fractions, we have

$$y\,dy = -x\,dx.$$

Each term of this equation may be integrated, giving

$$\frac{y^2}{2} = -\frac{x^2}{2} + C,$$

or

$$x^2 + y^2 = 2\,C. \qquad (1)$$

The value of C is determined by substituting $x = 3$, $y = 2$, in equation (1):

$$9 + 4 = 2\,C.$$

Hence $2\,C = 13$, and, on substitution of this value in equation (1), the final result becomes

$$x^2 + y^2 = 13.$$

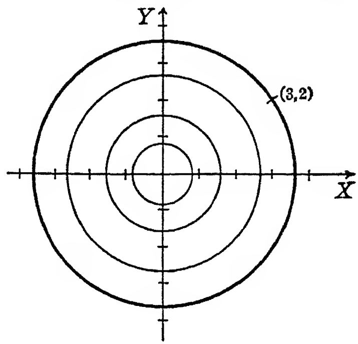


FIG. 133

Equation (1) represents the system of circles with center at the origin.

PROBLEMS

1. In the following problems find the equation of the curve which has the given slope at every point and which passes through the given point:

a. $\frac{dy}{dx} = \sqrt{x}$, (0, 0).

b. $\frac{dy}{dx} = \sqrt{y}$, (0, 0).

c. $\frac{dy}{dx} = \sqrt{4 - x}$, (0, 0).

d. $\frac{dy}{dx} = \frac{x}{y}$, (5, 0).

e. $\frac{dy}{dx} = \sqrt{xy}$, (1, 1).

f. $\frac{dy}{dx} = y^2$, (1, − 1).

g. $\frac{dy}{dx} = -\frac{4\,x}{9\,y}$, (0, 6).

h. $\frac{dy}{dx} = \frac{x}{\sqrt{y}}$, (1, 0).

i. $\frac{dy}{dx} = -3\,x$, (0, 2).

j. $\frac{dy}{dx} = \frac{-bx}{a\sqrt{a^2 - x^2}}$, (0, b).

k. $\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 4}}$, (0, 2).

l. $\frac{dy}{dx} = \frac{2}{3\sqrt[3]{x}}$, (0, 0).

m. $\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$, ($a$, 0).

n. $\frac{dy}{dx} = -\sqrt[3]{\frac{y}{x}}$, ($a$, 0).

2. In the following problems find the system of curves defined by the given equation. Draw three curves of each system.

a. $\frac{dy}{dx} = \frac{1}{2}$.

b. $\frac{dy}{dx} = \frac{4x}{y}$.

c. $\frac{dy}{dx} = -\frac{4x}{y}$.

d. $\frac{dy}{dx} = \frac{x}{2}$.

e. $\frac{dy}{dx} = \frac{1}{2y}$.

f. $\frac{dy}{dx} = x^2 - 1$.

g. $\frac{dy}{dx} = 3x^2$.

h. $\frac{dy}{dx} = \frac{4-x}{2+y}$.

i. $\frac{dy}{dx} = \frac{y^2}{x^2}$.

j. $\frac{dy}{dx} = \frac{x^2}{y^2}$.

k. $\frac{dy}{dx} = \frac{x}{y^2}$.

l. $\frac{dy}{dx} = \frac{x^2}{y}$.

3. Show that the curve whose slope is everywhere $3 - 2x$ and which passes through the point $(-2, 1)$ is a parabola, and draw its graph.

4. Show that the curve whose slope is everywhere $a + bx$ is a parabola.

5. At every point of a certain curve $y'' = -3$. Find the equation of this curve if it passes through the origin and has the slope 1 at this point.

6. At every point of a certain curve $y'' = \frac{24}{x^4}$. Find the equation of the curve if it passes through (2, 1) and has the slope -1 at that point.

7. At every point of a curve $y'' = 6x$, and it is tangent to the line whose equation is $2x - 3y = 6$ at the point $(0, -2)$. Find its equation.

8. Find the equation of the curve at every point of which $y'' = 15\sqrt{x}$ and which passes through the point (4, 0) with an inclination of 45°.

9. Find the equation of the curve at every point of which $y'' = \frac{6}{x^3}$ and which passes through the point (1, 1) with an inclination of 135°.

10. What is the system of curves determined by the equation $\frac{dy}{dx} = k$, where k is a constant?

11. Show that the equation $\frac{dy}{dx} = \frac{kx}{y}$ determines a system of hyperbolas or ellipses according as k is positive or negative.

12. Show that the equation $\frac{dy}{dx} = \frac{k}{y}$ determines a system of parabolas symmetrical with respect to the x-axis.

13. Show that the parabola is the only curve such that the rate of change of the slope with respect to x is constant.

148. Straight-Line Motion. If the distance s of a point moving along a straight line from a fixed point O is known as a function of the time t, the velocity and acceleration, as we have shown in Chapter VI, are obtained by differentiation. Thus

$$v = \frac{ds}{dt}$$

and

$$a = \frac{dv}{dt}.$$

It follows from these formulas that if the acceleration is known as a function of the time, the velocity is determined by integration. Thus

$$v = \int a\,dt.$$

And if the velocity is known as a function of the time, the distance is determined by integration. Thus

$$s = \int v\,dt.$$

Of course, each integration introduces an arbitrary constant, which can be determined if proper initial conditions are known.

Example 1. A particle moves with an acceleration of $\frac{1+t}{10}$ ft./sec.^{2}, its initial speed being 6 ft./sec. Find its speed and the distance moved after t seconds.

Solution. By the statement of the problem, the initial conditions are that when $t = 0$, $v = 6$; and we may choose the starting-point as the origin, so that $s = 0$.

Since $$a = \frac{1+t}{10},$$

$$v = \int \frac{1+t}{10}\,dt;$$

whence $$v = \frac{t}{10} + \frac{t^2}{20} + C. \tag{1}$$

If we substitute $t = 0$ and $v = 6$ in equation (1), we have $C = 6$, and

$$v = 6 + \frac{t}{10} + \frac{t^2}{20}. \tag{2}$$

But $$s = \int v\,dt, \quad \text{or} \quad s = \int \left(6 + \frac{t}{10} + \frac{t^2}{20}\right) dt.$$

Integrating, we have

$$s = 6t + \frac{t^2}{20} + \frac{t^3}{60} + C'. \tag{3}$$

The substitution of $t=0$ and $s=0$ in equation (3) gives $C'=0$, whence

$$s = 6t + \frac{t^2}{20} + \frac{t^3}{60}. \tag{4}$$

Equations (2) and (4) are the required results.

Example 2. A ball is thrown straight upward with a speed of 80 ft./sec. Neglecting the resistance of the air, find how high the ball will go.

Solution. It is assumed that the acceleration is due entirely to gravity. Let the positive direction of the line in which the ball moves be upward, with the origin at the point from which

the ball is thrown, and let the time be measured from the instant when the ball is thrown. Since the acceleration due to gravity is downward, it must be given the negative sign.

Hence $$a = -32 \text{ ft./sec.}^2$$

and $$v = \int -32\, dt.$$

Integrating, $$v = -32\, t + C. \tag{1}$$

The initial conditions for the determination of C are that $v = 80$ when $t = 0$. Substitution of these values in equation (1) gives $C = 80$. Hence

$$v = -32\, t + 80. \tag{2}$$

But $$s = \int v\, dt, \quad \text{or} \quad s = \int (-32\, t + 80)\, dt.$$

Integrating, we have $$s = -16\, t^2 + 80\, t + C'. \tag{3}$$

The initial conditions for the determination of C' are that $s = 0$ when $t = 0$. Substitution of these values in equation (3) gives $C' = 0$. Hence

$$s = -16\, t^2 + 80\, t. \tag{4}$$

As long as the motion continues without interruption the velocity is given by equation (2), and the position by equation (4). The ball will continue to rise until the velocity is zero. Setting $v = 0$ in equation (2) gives $t = 2.5$. Hence the ball moves upward for 2.5 sec. The height at this time is found by substituting $t = 2.5$ in equation (4), which gives $s = 100$. Therefore the ball will go to a height of 100 ft.

PROBLEMS

1. With what velocity will a stone strike the ground if dropped from the top of a building 120 ft. high?

2. With what velocity will the stone of Problem 1 strike the ground if thrown downward with a speed of 20 ft./sec.? if thrown upward with a speed of 20 ft./sec.?

3. A car makes a trip in 10 min., and its velocity is given by $v = 500t - 5t^3$, where t is measured in minutes and v in feet per minute. (*a*) How far does the car go? (*b*) What is its maximum speed? (*c*) How far has the car moved when its maximum speed is reached?

4. A stone dropped from a balloon, which was rising at the rate of 15 ft./sec., reached the ground in 8 sec. How high was the balloon when the stone was dropped?

5. In Problem 4, if the balloon had been falling at the rate of 15 ft./sec., how long would the stone have taken to reach the ground?

6. A train leaving a railroad station has an acceleration of $0.5 + 0.02t$ feet per second per second. Find how far it will have moved in 20 sec.

7. If the acceleration of a particle moving with a variable velocity v is $-kv^2$, where k is a constant, and if u is the initial velocity, show that $\frac{1}{v} = \frac{1}{u} + kt$.

HINT. Separate the variables before integrating.

8. If $v = 2$ when $t = 3$, find the relation between v and t, provided that the acceleration is

a. $2t - t^2$. *c.* $3t^{-\frac{3}{2}}$. *e.* $\sqrt{t} + 2$.

b. $\frac{1}{t^2} - t$. *d.* $4 - t^2$. *f.* $6 - \frac{1}{t^2}$.

9. Find the relation between s and t if $s = 2$ when $t = 1$ and if

a. $v = \sqrt{t - 1}$. *c.* $v = a + bt$.

b. $v = t^2 + \frac{1}{t^2}$. *d.* $v = t\sqrt{t^2 - 1}$.

10. A particle starts with an initial velocity u and is subject to a constant acceleration a. Show that the velocity and distance after time t are given by the formulas $v = u + at$ and $s = ut + \frac{1}{2}at^2$.

11. A particle sliding on a certain inclined plane is subject to an acceleration *downward* of 4 ft. per second per second. If it is started upward from the bottom of the plane with a velocity of 6 ft./sec., find the distance moved after t seconds. How far will it go before sliding backward?

12. If the inclined plane in Problem 11 is 20 ft. long, find the necessary initial speed in order that the particle may just reach the top of the plane.

149. Resolution of Forces. In the adjoining figure the arrow PT represents the magnitude and direction of a force F acting upon a particle at P. The quadrilateral $PRTS$ is a rectangle whose sides are parallel to the coördinate axes and whose diagonal is PT.

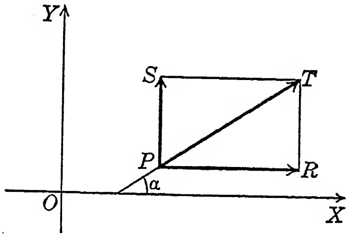


FIG. 134

In elementary mechanics we learn that the effect of the force F is the same as that of two forces acting parallel to the coördinate axes and of magnitudes equal to the sides of the rectangle constructed above. These forces are called the *components* of the force F. The *x-component* will be denoted by F_x and the *y-component* by F_y. Then $F_x = PR$ and $F_y = PS$.

By trigonometry,

$$PR = PT \cos \alpha \quad \text{and} \quad PS = PT \sin \alpha.$$

Hence $\quad F_x = F \cos \alpha \quad$ and $\quad F_y = F \sin \alpha.$

The process of replacing a force by its components is called *resolution of forces*, and is of great importance in solving problems involving forces.

150. The Suspension Cable. Suppose that we have a cable so loaded that the weight suspended (including the

weight of the cable) is w pounds per horizontal foot, w being constant. A suspension-bridge cable approximates this ideal condition. The problem is to determine the shape of the curve in which the cable hangs.

Let us take the origin at the bottom of the cable, with the x-axis horizontal. The part of the curve heavily marked represents a piece of the cable extending from the origin to some point P. The forces acting upon the piece OP are its weight, wx, and the tensions, H and T, at the ends of this piece. The line of action of H is horizontal and that of T is along the tangent to the curve at P (since the direction of a curve is the same as that of its tangent).

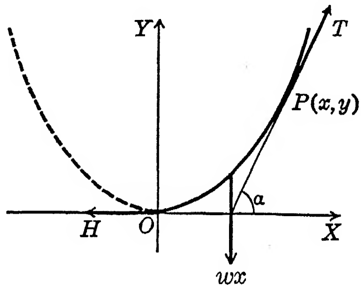


FIG. 135

Since the cable is in equilibrium, the horizontal and vertical components of T must be balanced by the forces H and wx respectively.

Hence $T \cos \alpha = H$ and $T \sin \alpha = wx$.

Dividing the second equation by the first, we have

$$\tan \alpha = \frac{wx}{H}.$$

But
$$\tan \alpha = \frac{dy}{dx}.$$

Hence
$$\frac{dy}{dx} = \frac{wx}{H},$$

or
$$dy = \frac{w}{H} x\, dx.$$

Integrating, $$y = \frac{w}{2H}x^2 + C.$$

Since $y = 0$ when $x = 0$, the constant $C = 0$. Thus the equation of the curve is

$$y = \frac{wx^2}{2H},$$

which is the equation of a parabola.

151. Curvilinear Motion. While an adequate study of curvilinear motion is beyond the scope of this book, the methods of § 148 can be extended to certain simple problems involving curvilinear motion.

We begin by defining acceleration along the x-axis as the time rate of change of the x-component (see § 89) of the velocity along the curve. Similarly, the acceleration along the y-axis is the time rate of change of the y-component of the velocity. If, then, the acceleration along the x-axis is some function of t, as $f(t)$, $\frac{dv_x}{dt} = f(t)$. One integration will give v_x as a function of t, and a second integration will give x as a function of t, since $\frac{dx}{dt} = v_x$. Likewise, we can find y as a function of t. The two equations thus found are called the *equations of motion* of the point, and are the parametric equations of the path pursued.

Example. A projectile is shot with an initial speed of 600 ft./sec. from a mortar inclined at an angle α whose sine is $\frac{3}{5}$. Find the path of the projectile, neglecting air resistance.

Solution. We first determine the initial conditions. Let us take the origin at the muzzle of the mortar and let the x-axis be horizontal. If $P(x, y)$ denotes the position of the projectile at time t, $x = 0$ and $y = 0$ when $t = 0$. From § 89 we

find that the x- and y-components of the initial velocity of 600 ft./sec. are $600 \cos \alpha = 600 \cdot \frac{4}{5} = 480$ and $600 \sin \alpha = 600 \cdot \frac{3}{5} = 360$. That is, when $t = 0$, $v_x = 480$ and $v_y = 360$.

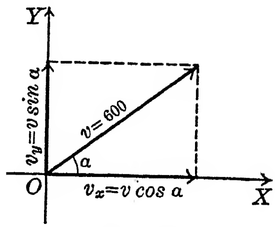


FIG. 136

Since air resistance is disregarded, $v_x = 480$ constantly.

Hence $$\frac{dx}{dt} = 480,$$

or $$dx = 480\, dt.$$

Integrating, $x = 480\,t + C$.

Since $x = 0$ when $t = 0$, $C = 0$, and we have

$$x = 480\,t. \tag{1}$$

The force of gravity, acting vertically, produces a negative acceleration of -32 ft. per second per second.

Hence $$\frac{dv_y}{dt} = -32, \quad \text{or} \quad dv_y = -32\,dt.$$

Integrating, $v_y = -32\,t + C$.

Since $v_y = 360$ when $t = 0$,

$$v_y = \frac{dy}{dt} = -32\,t + 360.$$

Integrating again, we have

$$y = 360\,t - 16\,t^2. \tag{2}$$

Equations (1) and (2) are the equations of motion required, or the parametric equations of the path. The curve can be easily plotted from these equations. If we eliminate the parameter t, we obtain the single equation

$$y = \frac{3\,x}{4} - \frac{x^2}{14400},$$

which shows that the path is a parabola with its axis vertical.

CHAPTER XI

DEFINITE INTEGRALS

152. The Definite Integral. Suppose that

$$\int f(x)\,dx = \phi(x) + C.$$

In this indefinite integral, $\phi(x) + C$, put first $x = b$, then $x = a$, and subtract. Thus

$$\phi(b) + C - [\phi(a) + C] = \phi(b) - \phi(a).$$

This result, from which the constant of integration has disappeared, is known as a definite integral.* It is indicated by the notation

$$\int_a^b f(x)\,dx,$$

which is read "the integral from a to b of $f(x)dx$." The number a is the lower limit, and b is the upper limit, of the integral.

Since the constant of integration does not appear in the final result, it is unnecessary to add it when finding the value of a definite integral. The operations involved in evaluating a definite integral as explained above are shown by the following notation:

(I) $$\int_a^b f(x)\,dx = [\phi(x)]_a^b = \phi(b) - \phi(a).$$

* In making this definition it is assumed that the function $f(x)$ varies continuously from $x = a$ to $x = b$.

Example 1. Find the value of $\int_{-1}^{2} x^2 dx.$

Solution. $\int_{-1}^{2} x^2 dx = \left[\frac{x^3}{3}\right]_{-1}^{2} = \left[\frac{8}{3}\right] - \left[-\frac{1}{3}\right] = 3.$

Example 2. Find the value of $\int_{-2}^{5} \frac{5\,x\,dx}{\sqrt{x^2+3}}.$

Solution.
$$\begin{aligned}\int_{-2}^{5} \frac{5\,x\,dx}{\sqrt{x^2+3}} &= 5\int_{-2}^{5} (x^2+3)^{-\frac{1}{2}}\,x\,dx.\\ &= \frac{5}{2}\int_{-2}^{5} (x^2+3)^{-\frac{1}{2}}\,2\,x\,dx\\ &= \left[5\sqrt{x^2+3}\right]_{-2}^{5}\\ &= 5\sqrt{28} - 5\sqrt{7}\\ &= 5\sqrt{7} = 13.23.\end{aligned}$$

PROBLEMS

Find the value of each of the following definite integrals:

1. $\int_{1}^{4} \frac{dx}{\sqrt{x}}.$

2. $\int_{2}^{6} \sqrt{x-2}\,dx.$

3. $\int_{0}^{4} x\sqrt{x^2+9}\,dx.$

4. $\int_{1}^{3} (x^2+x)\,dx.$

5. $\int_{0}^{3} x\sqrt{x^2+5}\,dx.$

6. $\int_{1}^{8} (x^{\frac{1}{2}} + x^{\frac{1}{3}})\,dx.$

7. $\int_{0}^{5} \sqrt{5\,x}\,dx.$

8. $\int_{0}^{2} \frac{x^2\,dx}{(10-x^3)^2}.$

9. $\int_{0}^{3} \sqrt{25-3\,x}\,dx.$

10. $\int_{1}^{3} x^2(2-3\,x)\,dx.$

11. $\int_{a}^{8\,a} (a^{\frac{2}{3}} - x^{\frac{2}{3}})\frac{dx}{x^{\frac{1}{3}}}.$

12. $\int_{1}^{4} \frac{dt}{\sqrt{5-t}}.$

13. $\int_0^a x\sqrt{a^2 - x^2}\,dx.$

14. $\int_{-3}^{-1} (s^2 - s^{-2})\,ds.$

15. $\int_1^4 \frac{4 - x^2}{x^4}\,dx.$

16. $\int_9^{16} x(1 + \sqrt{x})\,dx.$

17. $\int_2^3 s\sqrt{2s^2 - 5}\,ds.$

18. $\int_0^4 (4 - x)^3\,dx.$

19. $\int_0^a (\sqrt{a} - \sqrt{x})^2\,dx.$

20. $\int_0^2 z^2\sqrt{1 + z^3}\,dz.$

21. $\int_{-1}^0 (1 - t)(2 + t^2)\,dt.$

22. $\int_1^4 \frac{dx}{x^3}.$

23. $\int_0^a \sqrt{a - y}\,dy.$

24. $\int_1^4 \frac{1 + \sqrt{u}}{u^2}\,du.$

25. $\int_2^3 4\,dt.$

26. $\int_0^a \frac{x\,dx}{(x^2 + a^2)^2}.$

27. $\int_{-2}^1 x\sqrt{x + 3}\,dx.$

28. $\int_{-2}^2 \frac{x\,dx}{\sqrt{2x + 5}}.$

29. $\int_2^5 \frac{(x + 1)\,dx}{\sqrt{x - 1}}.$

30. $\int_4^1 \frac{4 - x^2}{x^4}\,dx.$

31. Prove that $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

32. A body moves with the speed $v = 4 - 6t$. Show that the distance moved between the times $t = 3$ and $t = 6$ is given by $\int_3^6 (4 - 6t)\,dt.$

153. Derivative of an Area. Let the equation of the curve LM of the figure be $y = f(x)$. Let BC be an ordinate at a fixed distance a from the y-axis and let DE be an ordinate at a variable distance x from the y-axis. It is apparent that the area $BCED$ is a function of x, which will be denoted by A. We have as yet no expression for A as a function of x, but with the aid of the figure we can

get the derivative of A with respect to x. For this purpose we carry out the fundamental process for finding a derivative; that is, we find the limiting value of $\frac{\Delta A}{\Delta x}$ when Δx approaches zero as a limit.

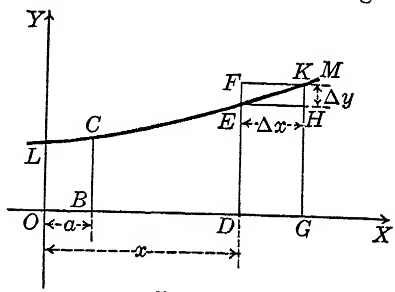


FIG. 137

Let x take on an increment Δx (DG in the figure); then A takes on a corresponding increment ΔA (the area $DEKG$ in the figure), and y takes on an increment Δy (HK in the figure).

Completing the rectangles $DEHG$ and $DFKG$, we see from the figure that

$$DEHG < \Delta A < DFKG.*$$

But $DEHG = y\Delta x$ and $DFKG = (y + \Delta y)\Delta x$;

hence $$y\Delta x < \Delta A < (y + \Delta y)\,\Delta x.$$

Dividing this by Δx, we have

$$y < \frac{\Delta A}{\Delta x} < y + \Delta y.$$

When Δx approaches zero as a limit, it is evident that $y + \Delta y$ approaches y as a limit, and hence $\frac{\Delta A}{\Delta x}$ must also approach y as a limit, since its value is always between y and $y + \Delta y$; that is,

$$\lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = y = f(x).$$

* If the curve LM were falling instead of rising, the inequality signs would be reversed.

But, by definition of a derivative,

$$\lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = \frac{dA}{dx}.$$

Hence

(II) $$\frac{dA}{dx} = f(x).$$

In words, *the derivative of the area bounded by the curve $y = f(x)$, the x-axis, a fixed ordinate, and a variable ordinate is equal to $f(x)$.*

If the differential notation is used, Formula II may be reduced at once to

(II *a*) $$dA = f(x)\,dx = y\,dx.$$

This may be remembered easily if it is noted that the differential $dA = y\,dx$ is merely the area of the rectangle $DEHG$ in the above figure. This is a close approximation to ΔA, which is the area of $DEKG$. (Compare § 87.)

154. Area under a Curve. Having formed the differential of the area described in the preceding paragraph, the area can be found by integration. Suppose it is required to find the area $PQRS$ bounded by the curve LM, the x-axis, the ordinate $PQ\,(x=a)$, and the ordinate $SR\,(x = b)$. Let the equation of LM be $y = f(x)$, and let A_x denote the area bounded by the curve LM, the x-axis, some fixed ordinate $PQ\,(x = a)$, and any other ordinate.

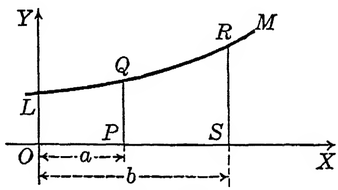


Fig. 138

In the preceding section we found that

$$dA_x = f(x)\,dx.$$

Hence, by integration,

$$A_x = \int f(x)\,dx = \phi(x) + C.$$

To determine the constant of integration we proceed exactly as in the previous chapter. Observe that $A_x = 0$, when $x = a$. Substituting these values, we find that $C = -\phi(a)$, and hence

$$A_x = \phi(x) - \phi(a).$$

The area $PQRS$ is obtained from this expression by setting $x = b$. Hence

$$\text{area } PQRS = \phi(b) - \phi(a).$$

But, by the definition of definite integral, this expression is the same as $\int_a^b f(x)\,dx$. Thus, if we denote the area $PQRS$ by A, we may write

$$\text{(III)} \qquad A = \int_a^b f(x)dx = \int_a^b y\,dx.$$

In words this may be stated thus: *The area bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = a$ and $x = b$ is given by the definite integral from a to b of $f(x)\,dx$.*

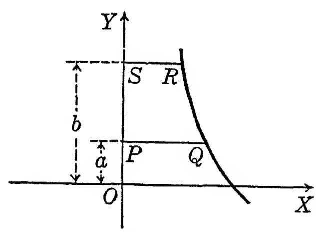


FIG. 139

The area discussed in this and the previous section is frequently referred to as "the area under the curve $y = f(x)$."

By an argument similar to the preceding we may establish the following statement:

The area bounded by the curve $x = f(y)$, the y-axis, and the lines $y = a$ and $y = b$ is given by

$$\int_a^b f(y)\,dy.$$

EXAMPLE 1. Find the area bounded by the line $y = x$, the x-axis, and the ordinates $x = 2$ and $x = 4$.

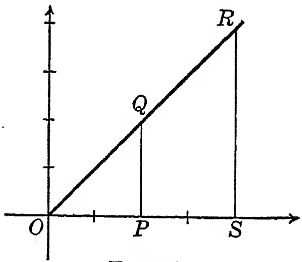


FIG. 140

Solution. The required area is given by

$$\int_2^4 x\,dx = \left[\frac{x^2}{2}\right]_2^4 = 8 - 2 = 6.$$

Hence the area is 6 square units.

This result may be verified by geometry, since $PQRS$ is a trapezoid in which one base is $PQ = 2$, the other base is $RS = 4$, and the altitude is $PS = 2$. Hence the area is $\frac{1}{2}(2 + 4) \times 2 = 6$.

EXAMPLE 2. Find the entire area bounded by the curve whose equation is $y^2 = x^2(4 - x^2)$.

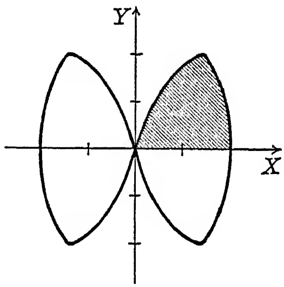


FIG. 141

Solution. Since the curve has symmetry with respect to both axes, it will be sufficient to find the area of the portion in the first quadrant and multiply this result by 4. To get the limits of integration, find the x-intercepts. These are 0 and 2.

Hence we have, by Formula III,

$$\frac{A}{4} = \int_0^2 y\,dx$$

$$= \int_0^2 x\sqrt{4 - x^2}\,dx.$$

Integrating this, we find

$$\frac{A}{4} = \left[-\frac{1}{3}(4 - x^2)^{\frac{3}{2}}\right]_0^2$$

$$= 0 + \tfrac{8}{3} = \tfrac{8}{3}.$$

Therefore $\qquad A = \frac{32}{3} = 10.67.$

PROBLEMS

In each of the following problems the result should be checked approximately by drawing the given curve carefully on coördinate paper and counting the number of squares in the required area.

1. Find the area bounded by the given curve, the x-axis, and the given ordinates:

a. $y = \frac{x^2}{4}$; $x = 0$, $x = 4$.

b. $y^2 = 2x$; $x = 2$, $x = 8$.

c. $y = 9x - x^3$; $x = 0$, $x = 3$.

d. $y = 8x - x^2$; $x = 2$, $x = 6$.

e. $y^2 = 4 - x$; $x = 0$, $x = 3$.

f. $y = \frac{10}{\sqrt{x+9}}$; $x = 0$, $x = 16$.

g. $y = x + \frac{4}{x^2}$; $x = 2$, $x = 4$.

h. $y = \frac{4x}{(x^2+1)^2}$; $x = 0$, $x = 3$.

i. $y^2 = 16 - 5x$; $x = -3$, $x = 0$.

j. $xy^2 = 12$; $x = 1$, $x = 64$.

k. $x^2y - x^2 + 9 = 0$; $x = 3$, $x = 12$.

l. $y^2 = x^3 + 4x^2$; $x = -4$, $x = 0$.

2. Find the area bounded by the given curve, the y-axis, and the given lines:

a. $x + y = 10$; $y = 2$, $y = 8$.

b. $y^2 = 2x$; $y = 1$, $y = 3$.

c. $y = \frac{x^2}{4}$; $y = 1$, $y = 4$.

d. $y^3 = x$; $y = 1$, $y = 2$.

3. Find the area bounded by the following curves and the x-axis:

a. $y = 4 - x^2$.

b. $y = 8 + 2x - x^2$.

c. $y = x - x^3$.

d. $y = x - \sqrt[3]{x}$.

e. $4y = x^4 - 8x^2$.

f. $x^2 = (y + 4)^3$.

4. Find the area bounded by the parabola $y^2 = 4x$ and the straight line $y = x$.

5. Find the area bounded by the two parabolas $y^2 = 8x$ and $x^2 = 8y$.

6. Find the area in the first quadrant bounded by the curve whose equation is $y = x^3$ and by the line whose equation is $y = 2x$.

7. Find the area bounded by the parabola whose equation is $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and by the x-axis and y-axis.

8. Find the area bounded by the parabola whose equation is $y^2 = 2px$ and by the chord through the focus perpendicular to the axis of the parabola.

9. Find the area bounded by the parabola whose equation is $x^2 = 2py$ and by the chord through the focus perpendicular to the axis of the parabola.

10. Show that the area of any segment of a parabola cut off by a chord perpendicular to the axis of the parabola is two thirds of the circumscribing rectangle.

11. Find the area of the segment of the parabola whose equation is $y = 6 + x - x^2$ cut off by the chord joining the points $(-1, 4)$ and $(3, 0)$.

12. Find in two ways the area bounded by the curve whose equation is $y^2 = 8 - 4x$ and the y-axis.

13. The velocity of a moving point is given by the formula $v = 8t - \sqrt{t}$. By determining the constant of integration, as in the previous chapter, show that the distance moved from time $t = 2$ to $t = 8$ is given by the definite integral

$$\int_2^8 (8t - \sqrt{t})\,dt,$$

and evaluate the integral.

14. Generalize Problem 13 by showing that if the velocity is $v = f(t)$, the distance moved from $t = a$ to $t = b$ is $\int_a^b v\,dt$. What *area* is equal to this distance?

155. Change of Limits in Integration by Substitution. In Chapter IX, § 145, we learned that certain expressions which cannot be integrated by the power formula can be

integrated by making a substitution of the type $\sqrt{a+bx}=v$. For example, to find $\int x\sqrt{9-x}\,dx$ we substitute $\sqrt{9-x}=v$, obtaining

$$\begin{aligned}\int x\sqrt{9-x}\,dx &= \int (2\,v^4 - 18\,v^2)\,dv \\ &= \tfrac{2}{5}\,v^5 - 6\,v^3 + C \\ &= \tfrac{2}{5}\,(9-x)^{\frac{5}{2}} - 6\,(9-x)^{\frac{3}{2}} + C.\end{aligned}$$

If the *definite integral* $\int_0^5 x\sqrt{9-x}\,dx$ is required, the work is shortened by changing the limits for x when substituting $\sqrt{9-x}=v$ to the *corresponding* limits for v. Thus, when $x=0$, $v=\sqrt{9-0}=3$; when $x=5$, $v=\sqrt{9-5}=2$.

Hence
$$\begin{aligned}\int_0^5 x\sqrt{9-x}\,dx &= \int_3^2 (2\,v^4 - 18\,v^2)\,dv \\ &= \left[\tfrac{2}{5}\,v^5 - 6\,v^3\right]_3^2 \\ &= \left[\tfrac{64}{5} - 48\right] - \left[\tfrac{486}{5} - 162\right] \\ &= \tfrac{148}{5}.\end{aligned}$$

This method enables us to avoid the troublesome substitution in terms of the original variable, and should always be employed.

PROBLEMS

1. Evaluate the following definite integrals:

a. $\displaystyle\int_0^a x\sqrt{a-x}\,dx.$

b. $\displaystyle\int_0^3 \frac{x^2\,dx}{\sqrt{x+1}}.$

c. $\displaystyle\int_2^7 \frac{x\,dx}{\sqrt{x+2}}.$

d. $\displaystyle\int_4^8 x\,(x-4)^{\frac{3}{2}}\,dx.$

e. $\displaystyle\int_{-7}^1 x\,\sqrt[3]{1-x}\,dx.$

f. $\displaystyle\int_0^a \frac{x\,dx}{\sqrt{x+a}}.$

2. Find the area inclosed by the loop of the curve whose equation is $4\,y^2 = x^2(4-x)$.

3. Find the area bounded by the curve whose equation is $y^2 = x^2(x^2 - 1)$ and by the line whose equation is $x = 2$.

4. Find the area inclosed by the loop of the curve whose equation is $y^2 = x^2(9 - x)$.

5. Find the area bounded by the curve whose equation is $y^2 = x^3 - x^2$ and by the line whose equation is $x = 2$.

6. Find the area inclosed by the loop of the curve whose equation is $y^2 = x(x - 2)^2$.

7. Find the area inclosed by the loop of the curve whose equation is $4y^2 = x^4(4 - x)$.

8. Write the integral giving the area bounded by the curve whose equation is $xy = 12$ and by the lines whose equations are $x = 1$ and $x = 5$. Can you evaluate this integral?

CHAPTER XII

THE DEFINITE INTEGRAL AS THE LIMIT OF A SUM

156. Introduction. In previous chapters an indefinite integral has been obtained by inverting the process of differentiation, and a definite integral has been defined as the difference between the two values of an indefinite integral corresponding to two values of the independent variable. It has been shown also that the area under a curve is given by a definite integral, from which it follows that any definite integral can be represented geometrically by the area under a curve.

In the historical development of the calculus a definite integral was first defined, from its geometrical representation, as the limiting value of a sum of terms of a certain form. The statement of this definition will be given below as a theorem. This theorem is very important for two reasons. In the first place, it enables us to make application of the integral calculus to many of the most important physical problems. The nature of these applications will be shown in the next chapter. In the second place, this theorem will enable us to use methods of approximation for calculating the value of a definite integral when the corresponding indefinite integral cannot be found. Two of these formulas for approximation will be given in the sections immediately following the theorem.

157. A Numerical Example. In order to get a clear understanding of the reasoning which leads to the theorem

concerning a definite integral as the limiting value of a sum of terms, let us consider first a simple numerical example. Suppose that it is required to find the area bounded by the curve $10y = 36 - x^2$, the x-axis, and the ordinates $x = 1$ and $x = 5$; that is, the area $ABCD$ of the figure below. The result is known by the method of the preceding chapter to be

$$\int_1^5 \frac{36 - x^2}{10}\,dx = \frac{1}{10}\left[36\,x - \frac{x^3}{3}\right]_1^5 = \frac{154}{15} = 10.267.$$

On the other hand, if the area $ABCD$ can be calculated in any way whatever, the result will be the value of the definite integral $\int_1^5 \frac{36 - x^2}{10}\,dx$. Thus the theorem of § 154 enables us to do two things: (1) to find the area under the curve $y = f(x)$ by means of the definite integral $\int_a^b f(x)\,dx$; (2) to find the value of a definite integral $\int_a^b f(x)\,dx$ by finding the value of the area under the curve $y = f(x)$.*

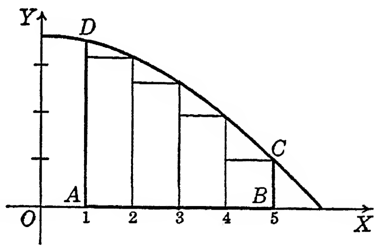


FIG. 142

A rough approximation to the required result may be found as follows: Let the segment AB be divided into a

*It may help to a realization of the importance of this statement and what follows if we consider the integral $\int_0^a \sqrt{a^2 - x^2}\,dx$. So far we have no formula for evaluating this. But by § 154 it is equal to the area bounded by the curve $y = \sqrt{a^2 - x^2}$, the x-axis, and the ordinates $x = 0$ and $x = a$. If we simplify the equation of the curve, we obtain $x^2 + y^2 = a^2$. Hence the geometrical representation of the given definite integral is that part of the circle $x^2 + y^2 = a^2$ which lies in the first quadrant. Its value, therefore, is $\frac{\pi a^2}{4}$.

certain number, say 4, of equal parts. The abscissas of the points of division will be $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 5$. At each point of division let ordinates to the curve be drawn, and with these ordinates as sides let rectangles be constructed as shown in the figure. Such a set of rectangles will be called an *inscribed set.* The sum of the areas of these rectangles will be an approximation to the area under the curve. The areas of the rectangles can be calculated. The base of each rectangle is 1; the altitudes, calculated from the equation of the curve, are as follows:

$$0.1\,(36 - 4) = 3.2. \qquad 0.1\,(36 - 16) = 2.0.$$
$$0.1\,(36 - 9) = 2.7. \qquad 0.1\,(36 - 25) = 1.1.$$

The sum of the areas of the four rectangles is 9.0.

A better approximation to the area $ABCD$ would be obtained by increasing the number of parts into which the segment AB is divided. Suppose that AB is divided into eight equal parts and rectangles inscribed as before. The base of each rectangle is 0.5; the altitudes are as follows:

$$0.1\,(36 - 2.25) = 3.375. \qquad 0.1\,(36 - 12.25) = 2.375.$$
$$0.1\,(36 - 4) = 3.200. \qquad 0.1\,(36 - 16) = 2.000.$$
$$0.1\,(36 - 6.25) = 2.975. \qquad 0.1\,(36 - 20.25) = 1.575.$$
$$0.1\,(36 - 9) = 2.700. \qquad 0.1\,(36 - 25) = 1.100.$$

The sum of the areas of the eight rectangles is 9.650.

If we continue to increase the number of parts into which AB is divided, it is apparent intuitively that the sum of the areas of the inscribed rectangles will approach the area $ABCD$ as a limit. The notation used to express this operation is the following: The interval AB is divided into n equal parts, the length of each part being denoted by Δx. The points of division are numbered 0, 1, 2, $\cdots$, n, the abscissas of these points being $x_0\,(=1)$, x_1, x_2, $\cdots$, $x_n\,(=5)$.

The altitude of the first rectangle is $0.1(36 - x_1^2)$, that of the second is $0.1(36 - x_2^2), \cdots$, and that of the nth is $0.1(36 - x_n^2)$. The sum of the areas is

$$0.1(36 - x_1^2)\Delta x + 0.1(36 - x_2^2)\Delta x + \cdots + 0.1(36 - x_n^2)\Delta x.$$

The usual notation for such a sum of terms is

$$\sum_{i=1}^{n} 0.1(36 - x_i^2)\Delta x,$$

read "the sum, from $i = 1$ to $i = n$, of $0.1(36 - x_i^2)\Delta x$." Since the area under the curve is given by the definite integral, it is apparent intuitively that

$$\lim_{n\to\infty}\sum_{i=1}^{n} 0.1(36 - x_i^2)\Delta x = \int_1^5 0.1(36 - x^2)\Delta x.$$

158. The General Case. Suppose that $y = f(x)$ is any function which is *continuous* in the interval from $x = a$ to $x = b$, which means that in this interval the graph of the function will have no breaks in it. In order to simplify the argument we will suppose also that $f(x)$ decreases constantly when x increases from a to b. The reasoning is the same if $f(x)$ increases constantly in the interval, the only difference being that, with the notation employed, the rectangles are circumscribed instead of inscribed. If $f(x)$ sometimes increases and sometimes decreases, the interval can be divided into smaller intervals within which $f(x)$ either increases or decreases constantly, and the reasoning can be applied to each of the smaller intervals separately.

Let us now consider the problem of finding the definite integral $\int_a^b f(x)\,dx$; that is, the area bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = a$ and $x = b$. A rough approximation to the required result may be found

as follows: Let the segment AB of the x-axis be divided into n equal parts and let the points of division, beginning with A, be numbered $0, 1, 2, \cdots, n$. The abscissas of the points of division will be denoted by $x_0(=a), x_1, x_2, \cdots x_n(=b)$. At each point of division let ordinates to the curve $y=f(x)$ be drawn, and with these ordinates as sides let rectangles be inscribed as shown in the figure. The sum of the areas of these rectangles will be an approximation to the area under the curve. The areas of the rectangles can be calculated. For if Δx denotes the base of each rectangle, we have

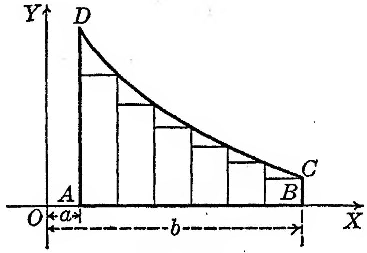


FIG. 143

$$\Delta x = \frac{b-a}{n}.$$

The altitude of the first rectangle is $f(x_1)$; of the second, $f(x_2)$; etc. Hence the sum of the areas of the n rectangles is

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x.$$

As n is increased indefinitely it is apparent intuitively that the sum of the areas of the inscribed rectangles approaches the area under the curve as a limit. Hence

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x.$$

The preceding construction may be generalized in two ways: (1) It is not necessary that the interval AB be divided into *equal* parts. (2) It is not necessary that the

altitudes of the rectangles should be equal to the ordinates at the points of division. This more general construction is made as follows: Let the segment AB be divided into n parts which are not necessarily equal. Let the length of the first part be Δx_1, of the second Δx_2, etc. Let x_1 be the abscissa of any point in the first part, x_2 the abscissa of any point in the second part, etc. With the ordinates at x_1, x_2, etc. as altitudes, construct rectangles as shown in the figure. Then the sum of the areas of these rectangles will be

$$\sum_{i=1}^{n} f(x_i)\,\Delta x_i.$$

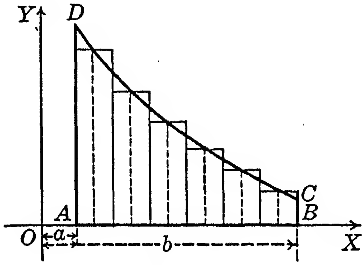


FIG. 144

If the length of each of the n parts into which AB is divided is made to approach zero when n increases without limit, the sum of the areas of the rectangles will approach the area under the curve as a limit. Consequently we shall have, as before,

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i.$$

159. The Fundamental Theorem. The last result of the preceding section has been established by using the geometrical interpretation of a definite integral. Every definite integral can be expressed geometrically as the area under a curve by simply plotting the curve $y = f(x)$, where $f(x)$ is the integrand. But in most applications the primary meaning of the definite integral is something quite different from an area, and it is convenient to state the result proved in the previous section as a theorem in analysis without

reference to its possible geometric representation. This statement is so important that it is usually referred to as the "Fundamental Theorem of the Integral Calculus."

THE FUNDAMENTAL THEOREM

Let $f(x)$ be a continuous function in the interval from $x = a$ to $x = b$.

Let this interval be divided into n subintervals Δx_1, Δx_2, $\ldots$, Δx_n.

Let x_i be any value of x in the subinterval Δx_i.

Then the limiting value of the sum

$$\sum_{i=1}^{n} f(x_i)\,\Delta x_i,$$

when n increases without limit and each subinterval approaches zero as a limit, is equal to the value of the definite integral

$$\int_a^b f(x)\,dx.$$

The conclusion of the above theorem may be expressed in symbols as follows:

$$\lim_{\substack{n \to \infty \\ \Delta x_i \to 0}} \sum_{i=1}^{n} f(x_i)\,\Delta x_i = \int_a^b f(x)\,dx.$$

The use of this theorem in applications will be illustrated in the next chapter. We shall now give two rules for calculating approximately the numerical value of a definite integral.

160. The Trapezoidal Rule. This rule is derived by approximating the area under a curve by means of trapezoids. The interval $b - a$ is divided into n parts, each of length Δx, and ordinates are constructed as before. The extremities of consecutive ordinates are then joined by straight

lines forming trapezoids. The abscissas of the points of division are $x_0 (= a), x_1, x_2, \cdots, x_n (= b)$. The corresponding values of $f(x)$ are $y_0 = f(x_0)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$, $\cdots$, $y_n = f(x_n)$. The areas of the trapezoids are given by the following expressions:

$$0.5\,(y_0 + y_1)\,\Delta x,$$

$$0.5\,(y_1 + y_2)\,\Delta x,$$

$$0.5\,(y_2 + y_3)\,\Delta x,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$0.5\,(y_{n-1} + y_n)\,\Delta x.$$

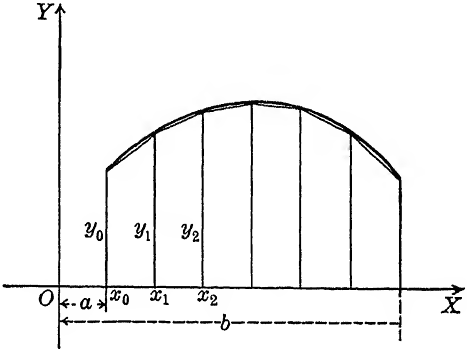


FIG. 145

The sum of the areas of the n trapezoids, which is obviously an approximation to the area under the curve, will be

$$(0.5\,y_0 + y_1 + y_2 + \cdots + y_{n-1} + 0.5\,y_n)\,\Delta x.$$

The method of procedure is formulated into a working rule as follows:

To evaluate approximately the integral $I = \int_a^b f(x)\,dx$, the interval $b - a$ is divided into n parts, each being equal to Δx. The abscissas of the points of division are $x_0 (= a)$, $x_1, x_2, \cdots, x_n (= b)$. The corresponding values of $f(x)$ are $y_0 = f(x_0)$, $y_1 = f(x_1)$, $\cdots$, $y_n = f(x_n)$. Then

(T) $\quad I = (0.5\,y_0 + y_1 + y_2 + \ldots + y_{n-1} + 0.5\,y_n)\,\Delta x.$

Exercise. Sketch the above figure and show that the trapezoids are equivalent to a certain set of rectangles; hence that the sum used in the trapezoidal rule is equal to a sum of the type

$$\sum_{i=1}^{n} f(x_i)\,\Delta x_i.$$

EXAMPLE. Using $n = 4$, compute the approximate value of

$$\int_0^2 \sqrt{4 + x^3}\, dx.$$

Solution. The table of values for x and y is first computed:

$x_0 = 0.0;\ y_0 = \sqrt{4+0}$
$= 2.000.$

$x_1 = 0.5;\ y_1 = \sqrt{4+0.125}$
$= 2.031.$

$x_2 = 1.0;\ y_2 = \sqrt{4+1}$
$= 2.236.$

$x_3 = 1.5;\ y_3 = \sqrt{4+3.375}$
$= 2.716.$

$x_4 = 2.0;\ y_4 = \sqrt{4+8}$
$= 3.464.$

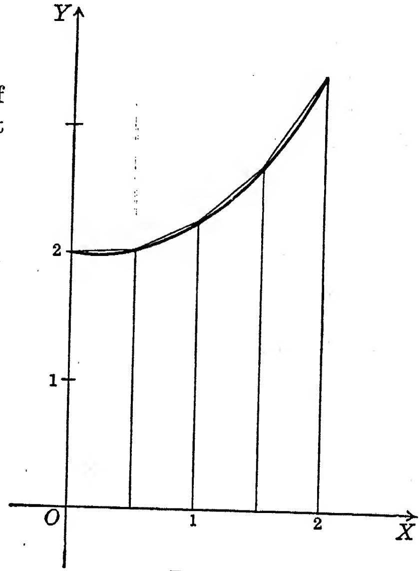


FIG. 146

Applying Formula T,

$I = (1.000 + 2.031 + 2.236 + 2.716 + 1.732) \times 0.5 = 4.858.$

(If we take $n = 10$, we obtain the more accurate value 4.826.)

PROBLEMS

1. A point moves with speed $v = 80 - 4t$. Show that the distance moved from $t = 2$ to $t = 8$ is given by $\int_2^8 (80 - 4t)\, dt$. Draw the graph of $v = 80 - 4t$, and calculate the value of the definite integral by finding the area under the graph.

2. Approximate the value of $\int_1^5 \frac{8\, dx}{x}$ by plotting $y = \frac{8}{x}$ on a large scale and finding the area under the curve (*a*) by counting squares, (*b*) by inscribing rectangles as in § 157, (*c*) by constructing rectangles as in the second figure of § 158.

3. Approximate the value of $\int_0^4 \frac{dx}{\sqrt{16+x^2}}$ by constructing a graph and estimating the area under it.

4. Compute the approximate values of the following integrals by the trapezoidal rule, using the values of n indicated:

a. $\int_0^6 \sqrt[3]{10+x^2}\,dx,\ n=3.$

b. $\int_0^4 \frac{dx}{\sqrt{1+x^3}},\ n=4.$

c. $\int_0^2 \sqrt{10-x^3}\,dx,\ n=4.$

d. $\int_0^5 \sqrt{125-x^3}\,dx,\ n=5.$

e. $\int_4^{10} \sqrt[3]{x^2-16}\,dx,\ n=3.$

f. $\int_0^{10} \sqrt{25+0.01x^4}\,dx,\ n=5.$

5. Compute the approximate values of the following integrals by the trapezoidal rule, using the values of n indicated. Check your results by performing the integrations.

a. $\int_0^3 x\sqrt{9-x^2}\,dx,\ n=6.$

b. $\int_1^4 \frac{x^2\,dx}{\sqrt{1+x^3}},\ n=3.$

161. Simpson's (Parabolic) Rule. For this rule it is necessary that n, the number of parts into which the interval $b-a$ is divided, shall be an *even* number. After the ordinates have been constructed as before, they are arranged in groups of three, the first group being y_0, y_1, y_2, the second y_2, y_3, y_4, etc. Through the extremities of

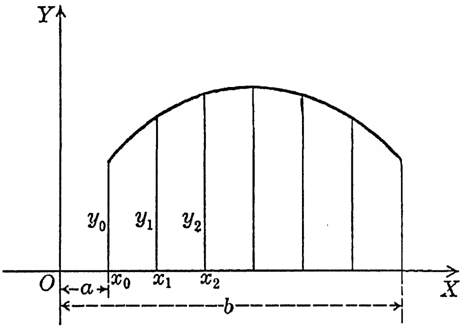


FIG. 147

the first group is passed an arc of a parabola having its axis parallel to the y-axis. The area under this parabola

between x_0 and x_2 is given by the formula (the proof of which is omitted)

$$\frac{\Delta x}{3}(y_0 + 4\,y_1 + y_2).*$$

Similarly, if a parabola is passed through the extremities of the next group of three ordinates, the area between x_2 and x_4 is

$$\frac{\Delta x}{3}(y_2 + 4\,y_3 + y_4).$$

The area under the parabola passed through the extremities of the last group of ordinates is

$$\frac{\Delta x}{3}(y_{n-2} + 4\,y_{n-1} + y_n).$$

By addition, the sum of the areas of the strips, which form an approximation to the area under the original curve, is

$$\frac{\Delta x}{3}(y_0 + 4\,y_1 + 2\,y_2 + 4\,y_3 + 2\,y_4 + \cdots + 4\,y_{n-1} + y_n).$$

The method of procedure is formulated into a working rule as follows:

To evaluate approximately the integral $I = \int_a^b f(x)\,dx$, the interval $b - a$ is divided into n (an even number of) parts, each being equal to Δx. The abscissas of the points of division are $x_0 (= a), x_1, x_2, \cdots, x_n (= b)$. The corresponding values of $f(x)$ are $y_0 = f(x_0), y_1 = f(x_1), \cdots, y_n = f(x_n)$. Then

$$\textbf{(S)}\quad I = (y_0 + 4\,y_1 + 2\,y_2 + 4\,y_3 + 2\,y_4 + \cdots + 4\,y_{n-1} + y_n)\frac{\Delta x}{3}.$$

*The proof of this formula is simple, although the algebraic work is tedious. The equation of a parabola with its axis parallel to the y-axis is $y = a + bx + cx^2$. Let $x_0 = h$, $x_1 = h + \Delta x$, and $x_2 = h + 2\Delta x$. The area beneath the parabola between the ordinates $x = h$ and $x = h + 2\,\Delta x$ can be found by integration. Calculating the values of y_0, y_1, and y_2 in terms of h and Δx, and substituting in the formula in the text, we find the same result as by the integration.

EXAMPLE. Using $n = 4$, compute the approximate value of

$$\int_0^2 \sqrt{4 + x^3}\,dx.$$

Solution. The table of values has been computed in § 160. Applying the formula S,

$$I = (2.000 + 8.124 + 4.472 + 10.864 + 3.464)\frac{0.5}{3} = 4.821.$$

PROBLEMS

1. Compute the approximate values of the following integrals by Simpson's rule, using the value of n indicated:

a. $\int_0^4 \sqrt[3]{10 + x^2}\,dx$, $n = 4$.

b. $\int_0^4 \sqrt{100 - x^3}\,dx$, $n = 4$.

c. $\int_0^2 \sqrt{9 + x^4}\,dx$, $n = 4$.

d. $\int_4^{10} \sqrt[3]{x^2 - 16}\,dx$, $n = 6$.

2. In the following problems compute the approximate values of the integrals by both the trapezoidal and Simpson's rules. If the indefinite integral can be found, calculate also the exact value of the integral.

a. $\int_0^6 \sqrt{36 - x^2}\,dx$, $n = 6$.

b. $\int_1^5 \frac{dx}{x}$, $n = 8$.

c. $\int_0^4 (16 - x^2)\,dx$, $n = 8$.

d. $\int_0^6 x\sqrt{36 - x^2}\,dx$, $n = 6$.

e. $\int_1^5 \frac{dx}{x^2}$, $n = 8$.

f. $\int_2^6 \frac{dx}{\sqrt{x^2 - 1}}$, $n = 4$.

g. $\int_2^6 \frac{x\,dx}{\sqrt{x^2 - 1}}$, $n = 4$.

3. A field is bounded by two parallel straight lines and two irregular curves. The length between the straight sides is 50 rd. Starting at one end, the width parallel to the straight sides is measured at 5-rod intervals with the results 35, 37, 35, 32, 30, 31, 30, 28, 25, 21, and 12 rd. Find the area in acres by Simpson's rule. (1 acre = 160 sq. rd.)

CHAPTER XIII

APPLICATIONS OF THE FUNDAMENTAL THEOREM

162. Introduction. The Fundamental Theorem of the Integral Calculus states that, under proper conditions, the limiting value of a sum of terms, each of the form $f(x_i)\Delta x_i$, as the number of terms is increased indefinitely, is equal to the definite integral, between proper limits, of $f(x)\,dx$. The present chapter is devoted to some of the many applications of this theorem. The solution of each problem consists of two distinct steps: (1) *setting up the integral*, which means the determination of the function $f(x)$ and of the limits of integration; (2) the evaluation of the integral.

Whenever possible, a definite integral should be evaluated by finding the indefinite integral and substituting the limits. When the indefinite integral cannot be found by the formulas which are available, the value of the definite integral may be approximated by the trapezoidal and Simpson's rules.

163. Plane Areas. It has already been shown that the area bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = a$ and $x = b$ is given by the definite integral $\int_a^b f(x)\,dx$. For more general cases, in which the required area is not bounded by a coördinate axis and by straight lines perpendicular to this axis, it is usually convenient to make use of the Fundamental Theorem as illustrated in the following examples. In all such problems a correct figure is absolutely necessary.

EXAMPLE 1. Find the area bounded by the parabola $y^2 = 2x$ and the straight line $2y = x$.

First Solution. The curves intersect at the origin and at (8, 4). The area to be found is shown in the figure. An approximate value of the area may be obtained by dividing it into n narrow strips, by lines parallel to the y-axis, and then replacing each strip by a rectangle. The figure represents an enlargement of strip number i, of width Δx_i. The area of this strip is approximated by that of a rectangle having the same width Δx_i and a length equal to the difference between the ordinate of the parabola and that of the straight line at some point x_i within the interval. At this point the ordinate of the parabola is $\sqrt{2x_i}$, and the ordinate of the straight line is $\frac{x_i}{2}$. Hence the length of the rectangle is $\sqrt{2x_i} - \frac{x_i}{2}$, and the area of the rectangle is

$$\left(\sqrt{2x_i} - \frac{x_i}{2}\right)\Delta x_i.$$

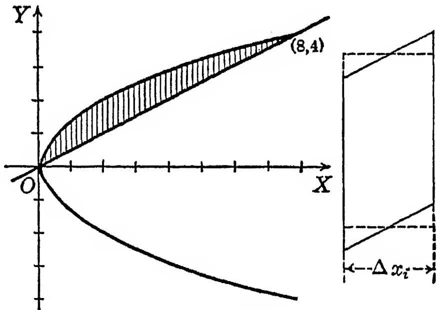


FIG. 148

Now the important point is that this formula applies to *every* strip into which the required area has been divided; for the lower end of *every* strip is on the straight line, and the upper end is on the parabola. Hence an approximate value of the area will be

$$\sum_{i=1}^{n}\left(\sqrt{2x_i} - \frac{x_i}{2}\right)\Delta x_i,$$

and, by the Fundamental Theorem, the exact value will be

$$A = \int_0^8 \left(\sqrt{2x} - \frac{x}{2}\right)dx,$$

the limits being assigned to include all the strips. Integrating,

$$A = \left[\frac{1}{3}(2x)^{\frac{3}{2}} - \frac{x^2}{4}\right]_0^8 = \frac{64}{3} - \frac{64}{4} = \frac{16}{3}.$$

The detailed reasoning given in the solution above must be clearly understood in order to avoid mistakes. But not all the details need be reproduced in every solution. The area of one of the small rectangles is called an *element of area* and is denoted by dA. In writing this element of area we may omit the subscript i and replace Δx_i by the differential dx, which gives at once the expression to be integrated. The essential details will be illustrated by a second solution.

Second Solution. Let the area be divided into strips by lines parallel to the x-axis. One of these strips is shown in the figure. Every such strip will have its left-hand end on the parabola and its right-hand end on the straight line. The width of a strip is dy, and its length is $x_1 - x_2$, where x_1 is the abscissa of a point on the line (since this is the larger one) and x_2 is the abscissa of a point on the parabola. Hence the element of area is

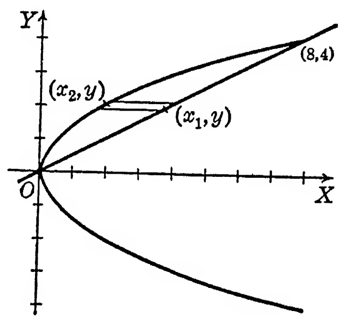


FIG. 149

$$dA = (x_1 - x_2)\,dy = \left(2y - \frac{y^2}{2}\right)dy,$$

by substitution from the equations of the curves. When all the strips are included, y varies from 0 to 4. Hence

$$A = \int_0^4 \left(2y - \frac{y^2}{2}\right)dy = \left[y^2 - \frac{y^3}{6}\right]_0^4 = 16 - \frac{32}{3} = \frac{16}{3}.$$

The result may be checked by drawing the figure on squared paper and counting the squares within the area.

It has appeared in the preceding example that the elementary strips may be taken parallel either to the x-axis or the y-axis. This freedom of choice does not exist in all problems, as shown in the next example.

When the strips are taken parallel to the y-axis, the element of area is expressed in terms of x and dx. In this case it is said that we *integrate with respect to x.* When the strips are taken parallel to the x-axis, we *integrate with respect to y.* The elements of area are respectively equal to

$$dA = (y_1 - y_2)\,dx$$

and

$$dA = (x_1 - x_2)\,dy.$$

EXAMPLE 2. Find the area bounded by the parabola $y^2 = 2\,x$ and the straight line $x - y = 4$.

Solution. The curves intersect at $(2, -2)$ and $(8, 4)$.

If the area is divided into strips parallel to the x-axis, every strip will have its left-hand end on the parabola and its right-hand end on the straight line. It is possible then to write *one* formula which will give the area of *every* strip, and thus to apply the Fundamental Theorem.

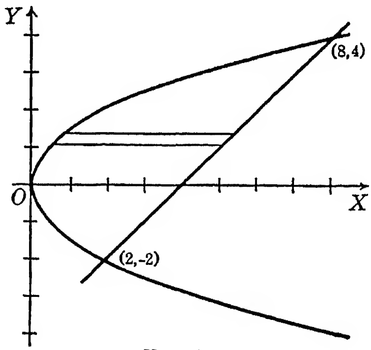


FIG. 150

If the area is divided into strips parallel to the y-axis, the strips between $x = 0$ and $x = 2$ will extend from the lower to the upper branch of the parabola, while those between $x = 2$ and $x = 8$ will extend from the straight line to the parabola. In this case it is not possible to write one formula which will give the area of every strip, and the area cannot be expressed by a single integral.

In order to express the area by a single integral, it is necessary to integrate with respect to y. The element of area is

$$dA = (x_1 - x_2)\,dy,$$

where x_1 is the abscissa of a point on the line and where x_2 is the abscissa of a point on the curve. Substitution from the equations of the line and the curve gives

$$dA = \left(y + 4 - \frac{y^2}{2}\right) dy.$$

When all the strips are included, y varies from -2 to 4. Hence

$$A = \int_{-2}^{4} \left(y + 4 - \frac{y^2}{2}\right) dy = \left[\frac{y^2}{2} + 4y - \frac{y^3}{6}\right]_{-2}^{4}$$
$$= (8 + 16 - \tfrac{32}{3}) - (2 - 8 + \tfrac{4}{3}) = 18.$$

PROBLEMS

1. Find the areas bounded by the following curves. In each case draw the figure, showing the element of area.

a. $y = x^3,\ y = 4x$.
b. $y = 6x - x^2,\ y = x$.
c. $x = 4y - y^2,\ y = x$.
d. $y = 4x - x^2,\ y = 2x - 3$.
e. $y^2 = 4x,\ 2x - y = 4$.
f. $y = x^3 - 3x,\ y = x$.
g. $y = x^2,\ 2x - y + 3 = 0$.
h. $y = x^2 - 2x - 3,\ y = 6x - x^2 - 3$.
i. $y^2 = 4x,\ x = 12 + 2y - y^2$.
j. $y = x^3 - x,\ y = x - x^2$.

2. Find the area included between the two parabolas $y^2 = 2px$ and $x^2 = 2py$.

3. Find the area included between the two parabolas $y^2 = ax$ and $x^2 = by$.

4. Find the entire area of the curve $y^2 = 9x^2 - x^4$.

5. Find the area bounded by the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and the coördinate axes.

6. Find the area bounded by the curve whose equation is $y = 6 - x^2$ and the line whose equation is $y = -3$.

7. Find the area bounded by the curve whose equation is $y = 6 + 4x - x^2$ and the chord joining $(-2, -6)$ and $(4, 6)$.

8. Find the area bounded by the curve whose equation is $y^3 = x^2$ and the chord joining $(-1, 1)$ and $(8, 4)$.

9. Find the area bounded by the curves whose equations are $y^2 = x$, $y = x$, and $2y = x$.

10. Find the area under the hyperbola $xy = 24$ from $x = 4$ to $x = 8$ (use Simpson's rule, taking $n = 8$).

11. Write the integral giving the area of a segment of a circle of radius 10 cut off by a chord distant 6 from the center. Approximate the integral by Simpson's rule, taking $n = 4$.

12. Find the area bounded by the curve whose equation is $y = x(1 \pm \sqrt{x})$ and the line whose equation is $x = 4$.

13. Find the entire area of the loop of the curve whose equation is $y^2 = 4x^2 - x^3$.

14. Find the area bounded by the curve whose equation is $x^2y = x^2 - 1$ and the lines whose equations are $y = 1$, $x = 1$, and $x = 4$.

15. Find the area bounded by the curve whose equation is $y = x^3 - 9x^2 + 24x - 7$, the y-axis, and the line whose equation is $y = 29$.

16. Find the area of a circle of radius 6 by dividing it into elements of area which are concentric rings of width Δr.

17. Plot on the same coördinate axes the circle whose equation is $x^2 + y^2 = a^2$ and the ellipse whose equation is $b^2x^2 + a^2y^2 = a^2b^2$. Set up the integrals giving the areas within the curves and, without integrating, show that the area of the ellipse is $\frac{b}{a}$ times the area of the circle, and therefore is πab.

164. Length of a Curve. Let AB be a curve whose equation in rectangular coördinates is $y = f(x)$. Let the interval $b - a$ be divided into n parts and let ordinates to the curve be erected at each point of division. When the extremities of these ordinates are joined by straight lines, we have a system of chords forming a broken line whose length may be utilized to approximate the length of the curve. Moreover, when the number of points of division is increased indefinitely in such a way that the length of

each chord approaches zero as a limit, the length of the broken line approaches the length of the curve AB as a limit.

Let s denote the length of AB, and let Δs_i denote the length of one of the chords. Then the length of the broken line is $\sum_{i=1}^{n}\Delta s_i$, and

$$s = \lim_{\substack{n\to\infty \\ \Delta s_i \to 0}} \sum_{i=1}^{n}\Delta s_i. \quad (1)$$

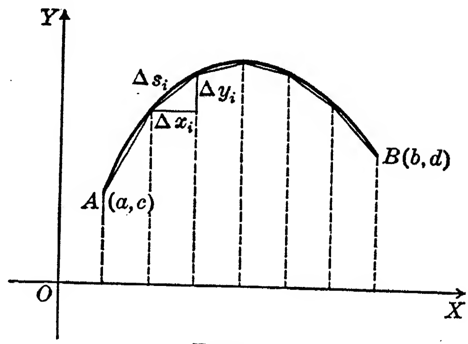


FIG. 151

In order to apply the Fundamental Theorem to this, we must express Δs_i in the form $f(x_i)\Delta x_i$. We do this in the following manner:

Each chord Δs_i is the hypotenuse of a right triangle, the sides of which are, in the increment notation of the differential calculus, Δx_i and Δy_i. Hence

$$\Delta s_i = \sqrt{\overline{\Delta x_i}^2 + \overline{\Delta y_i}^2}.$$

If we factor out Δx_i, this becomes

$$\Delta s_i = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i. \quad (2)$$

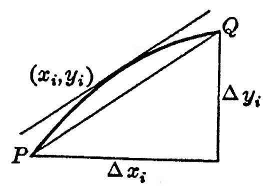


FIG. 152

The quotient $\frac{\Delta y_i}{\Delta x_i}$ is the slope of the chord PQ. It is apparent from the figure that there is a point (x_i, y_i) on the arc PQ at which the tangent to the curve is parallel to the chord PQ. Hence, if $\left(\frac{dy}{dx}\right)_{x=x_i}$ denotes the value of the derivative for $x = x_i$, we have

$$\frac{\Delta y_i}{\Delta x_i} = \left(\frac{dy}{dx}\right)_{x=x_i}$$

Thus equation (2) becomes

$$\Delta s_i = \sqrt{1+\left(\frac{dy}{dx}\right)^2_{x=x_i}}\,\Delta x_i. \tag{3}$$

This is of the form $f(x_i)\,\Delta x_i$.

When $\Delta s_i \rightarrow 0$, $\Delta x_i \rightarrow 0$ also. Hence the substitution of (3) in (1) gives

$$s = \lim_{\substack{n\to\infty \\ \Delta x_i \to 0}} \sum_{i=1}^{n} \sqrt{1+\left(\frac{dy}{dx}\right)^2_{x=x_i}}\,\Delta x_i.$$

Applying the Fundamental Theorem, we obtain the following formula:

(I) $$s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

Another form for the integral may be obtained by dividing the interval $d-c$ on the y-axis into n parts and factoring Δy_i out of the radical in equation (2). The result is

(II) $$s = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy.$$

An easy way of remembering both forms is to regard the element of arc ds as a chord which is the hypotenuse of a small right triangle whose sides are dx and dy. Then

$$ds = \sqrt{(dx)^2+(dy)^2}.$$

Factoring out dx, we obtain Formula I; factoring out dy, we obtain Formula II.

Example. Find the length of the curve $9y^2 = 4(1+x^2)^3$ from the point where $x = 0$ to the point where $x = 3$.

Solution. From the equation of the curve,

$$y = \tfrac{2}{3}(1+x^2)^{\frac{3}{2}}.$$

Differentiating,

$$\frac{dy}{dx} = 2x\sqrt{1+x^2}.$$

Using Formula I,

$$ds = \sqrt{1+4x^2(1+x^2)}\,dx = (1+2x^2)\,dx.$$

Hence $$s = \int_0^3 (1+2x^2)\,dx = \left[x + \frac{2x^3}{3}\right]_0^3 = 21.$$

Gross errors may usually be detected by comparing the result with the length of the chord joining the end points of the curve. In this problem the end points are (0, 0.67) and (3, 21.1); hence the length of the chord is 20.6.

PROBLEMS

1. Find the length of the curve whose equation is $y^2 = x^3$ from the point where $x = 0$ to the point where $x = 5$.

2. Calculate the length of the curve whose equation is $9y^2 = (2+x^2)^3$ from the point where $x = 0$ to the point where $x = 2$.

3. Find the length of the curve whose equation is $y^3 = ax^2$ between the points (0, 0) and (a, a).

4. Find the length of that part of the curve whose equation is $(y-8)^2 = x^3$ which is intercepted between the coördinate axes.

5. Find the entire length of the hypocycloid whose equation is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

6. Find the length of the curve whose equation is $y^3 = x^2$ between the points (0, 0) and (8, 4).

7. Find the length of the curve whose equation is $y = \frac{x^3}{6} + \frac{1}{2x}$ from the point where $x = 1$ to the point where $x = 3$.

8. Find by integration the length of the straight line whose equation is $y = mx + b$ between the points $(0, b)$ and (x, y). Check your result by geometry.

In the following problems the indefinite integrals cannot be found by the rules previously given. The values of the definite integrals must be approximated.

9. Approximate the length of the parabola whose equation is $y = x^2$ between (0, 0) and (2, 4). Use Formula I for ds, and evaluate the integral by Simpson's rule, taking $n = 4$.

10. Approximate the length of the parabola whose equation is $y^2 = 2x$ between (0, 0) and (8, 4). Use Formula II for ds, and evaluate the integral by Simpson's rule, taking $n = 4$.

11. Approximate the length of the curve whose equation is $y = x^3$ between (0, 0) and (2, 8). Use Formula I for ds, and evaluate the integral by Simpson's rule, taking $n = 4$.

12. Approximate the length of the hyperbola whose equation is $x^2 - y^2 = 9$ from (3, 0) to (5, 4). Use Formula II for ds, and evaluate the integral by the trapezoidal rule, taking $n = 4$.

13. Approximate the length of the arch of the parabola whose equation is $y = 4x - x^2$ which lies above the x-axis.

165. Volumes of Solids of Revolution. A solid of revolution is a solid having an axis of symmetry such that every section of the solid made by a plane perpendicular to this axis is a circle with its center on the axis. When a plane area is revolved about an axis in its plane, it is said to *generate* a solid of revolution. For example, a right circular cone is generated by revolving a right triangle about one of its legs, and a sphere is generated by revolving a semicircular area about its diameter.

Let the area bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = a$ and $x = b$ be revolved about the x-axis, forming a solid of revolution. To find the volume of such a solid we proceed as in § 163. Let the segment $b - a$ of the x-axis be divided into n parts, and let planes perpendicular to the x-axis be passed through each point of division. These section planes divide the solid into n circular plates.

The volume of any one of these circular plates will be approximately equal to that of a cylinder having the same thickness Δx and a radius, r, equal to the radius of the plate at some point within the corresponding subinterval Δx.* The cylinder would be generated by the revolution of a rectangle, as shown in Fig. 154. The volume of such a cylinder is $\pi r^2 \Delta x$. An approximation to the volume of the solid is then given by the sum of the volumes of the small cylinders; that is, by $\sum \pi r^2 \Delta x$, the summation extending over all the subintervals Δx of the interval $b-a$.

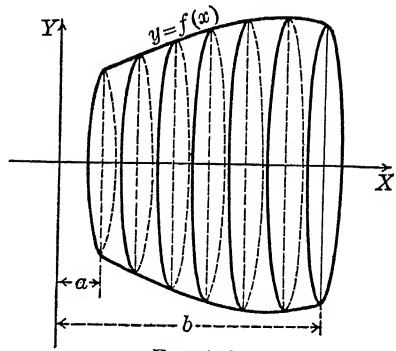


FIG. 153

In the present problem the radius r is equal to an ordinate to the curve $y=f(x)$; that is, $r=y=f(x)$. Hence, by the Fundamental Theorem, the exact volume of the solid is given by

$$\lim \sum \pi [f(x)]^2 \Delta x = \int_a^b \pi [f(x)]^2 dx.$$

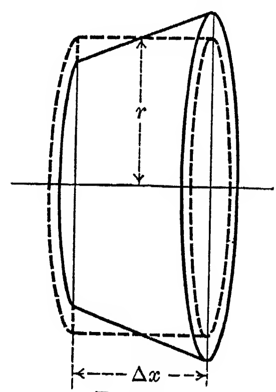


FIG. 154

This formula is valid only for the particular type of problem described; that is, when the area under the curve $y=f(x)$ is revolved about the x-axis. The general method can be applied to any case of a solid of revolution, and should

*In this and the following applications the subscripts hitherto used will be omitted for the sake of simplifying the notation.

always be used without attempting to memorize formulas. The essential steps in the general method are the following:

I. Draw the figure showing an element of volume, a right cylinder. Denote the thickness of the element by dx if the axis of revolution is parallel to the x-axis (and by dy if the axis of revolution is parallel to the y-axis).

II. Let r denote the radius of the element of volume. With the aid of the figure express r in terms of x (or y) from the equation of the given curve.

III. Form the element of volume

$$dV = \pi r^2 dx \text{ (or } \pi r^2 dy\text{)}.$$

IV. Determine the limits of integration from the figure.

The following examples illustrate the application of the general method to the different types of problems. These examples are based on the accompanying figure, in which the equation of the curve OA is $y^2 = 2x$. The letters x and y as used in the problems below always represent the coördinates of a point on the curve OA.

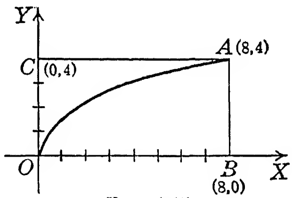


FIG. 155

EXAMPLE 1. Find the volume generated by revolving the area OAB about the x-axis.

Solution. An element of volume is shown in the figure. The thickness is dx, and the radius is an ordinate of the curve.

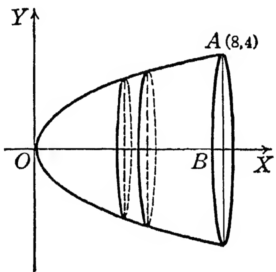


FIG. 156

Hence $\quad r = y = \sqrt{2x}$

and $\quad dV = \pi r^2 dx = \pi\, 2x dx.$

$$V = \int_0^8 2\pi x dx = \pi\left[x^2\right]_0^8 = 64\pi = 201.1.$$

Gross errors may often be detected by comparing the result obtained with the volume of an inscribed or circumscribed cone or cylinder. In this example the volume of the inscribed cone is $\frac{128\,\pi}{3}$ and that of the circumscribed cylinder is $128\,\pi$.

EXAMPLE 2. Find the volume generated by revolving the area OAB about the line AB.

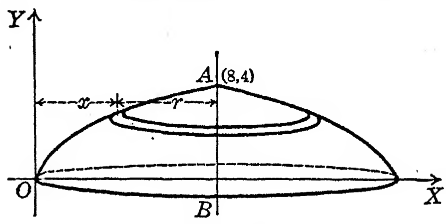


FIG. 157

Solution. An element of volume is shown in the figure. The thickness is dy, and the radius is $8 - x$.

Hence
$$r = 8 - x = 8 - \frac{y^2}{2}$$

and
$$dV = \pi r^2 dy = \pi\left(64 - 8\,y^2 + \frac{y^4}{4}\right)dy.$$

$$V = \int_0^4 \pi\left(64 - 8\,y^2 + \frac{y^4}{4}\right)dy = \pi\left[64\,y - \frac{8\,y^3}{3} + \frac{y^5}{20}\right]_0^4$$

$$= \frac{2048\,\pi}{15} = 428.9.$$

EXAMPLE 3. Find the volume generated by revolving the area OAC about the y-axis.

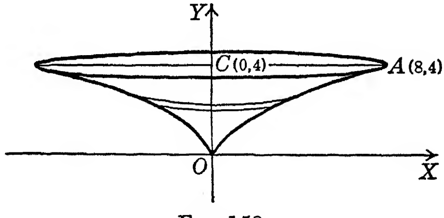


FIG. 158

Solution. An element of volume is shown in the figure. The thickness is dy, and the radius is an abscissa of the curve.

Hence $$r = x = \frac{y^2}{2}$$

and $$dV = \pi r^2 dy = \frac{\pi}{4} y^4 dy.$$

$$V = \int_0^4 \frac{\pi}{4} y^4 dy = \frac{\pi}{4}\left[\frac{y^5}{5}\right]_0^4$$

$$= \frac{256\,\pi}{5} = 160.8.$$

EXAMPLE 4. Find the volume generated by revolving the area OAC about the line CA.

Solution. An element of volume is shown in the figure. The thickness is dx and the radius is $4 - y$.

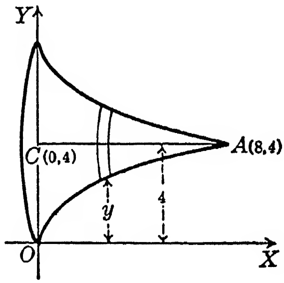


FIG. 159

Hence

$$r = 4 - y = 4 - \sqrt{2x}$$

and $$dV = \pi r^2 dx$$

$$= \pi(16 - 8\sqrt{2x} + 2x)dx.$$

$$V = \int_0^8 \pi(16 - 8\sqrt{2x} + 2x)\,dx$$

$$= \pi\left[16x - \tfrac{8}{3}(2x)^{\frac{3}{2}} + x^2\right]_0^8$$

$$= \frac{64\,\pi}{3} = 67.02.$$

166. Simplification by a Change of Variable. In Example 4, above, the desired volume is given by the integral

$$V = \pi\int_0^8 (4 - y)^2 dx.$$

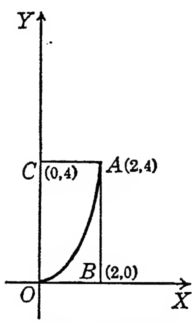
Instead of substituting for y its value in terms of x, we may change the variable of integration by substituting for dx in terms of y and dy.

Thus, since $y^2 = 2x$, $2y\,dy = 2\,dx$, or $dx = y\,dy$. When $x = 0$, $y = 0$; when $x = 8$, $y = 4$. Substituting and changing the limits, we have

$$V = \pi \int_0^8 (4 - y)^2 dx = \pi \int_0^4 (4 - y)^2 y\,dy$$
$$= \pi \left[8y^2 - \frac{8}{3}y^3 + \frac{y^4}{4} \right]_0^4$$
$$= \frac{64\pi}{3} \text{ (as before).}$$

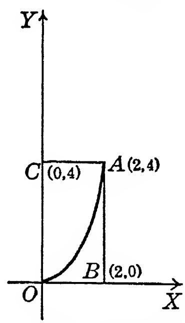
This process is merely integration by an appropriate substitution, similar to that discussed in §§ 145 and 155. It frequently enables one to avoid troublesome fractional exponents.

PROBLEMS



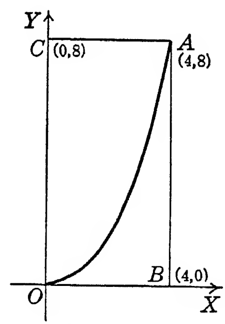
Equation of OA is $y = x^2$

FIG. 160



Equation of OA is $2y = x^3$

FIG. 161



Equation of OA is $y^2 = x^3$

FIG. 162

1. In Fig. 160 above find the volume generated when the area

a. OAB is revolved about OX.
b. OAB is revolved about AB.
c. OAB is revolved about CA.
d. OAB is revolved about OY.
e. OAC is revolved about OY.
f. OAC is revolved about CA.
g. OAC is revolved about AB.
h. OAC is revolved about OX.

2. The same as Problem 1 for Fig. 161.

3. The same as Problem 1 for Fig. 162.

4. Find by integration the volume of the cone generated by revolving about the x-axis the triangle whose vertices are $(0, 0)$, $(a, 0)$, (a, b).

5. Find by integration the volume of the cone generated by revolving about the y-axis the triangle whose vertices are $(0, 0)$, (a, b), $(0, b)$.

6. Find the volume of the paraboloid of revolution generated by revolving about the x-axis the area bounded by $y^2 = 2\,px$ and $x = h$. Show that the result is one half the volume of the cylinder having the same base and altitude.

7. Find the volume generated by revolving about the y-axis the area bounded by $y^2 = 2\,px$, $y = b$, and the y-axis. Show that the result is one fifth the volume of the cylinder having the same base and altitude.

8. Find the volume of the oblate spheroid generated by revolving the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the y-axis.

9. Find the volume of the prolate spheroid generated by revolving the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the x-axis.

10. A hyperboloid of revolution is generated by revolving the hyperbola $x^2 - y^2 = a^2$ about the x-axis. Find the volume of a segment of one base of this solid of thickness a.

11. Find the volume generated by revolving about the x-axis the area of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

12. Find the volume generated by revolving about the x-axis the area bounded by the coördinate axes and the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

13. Find the volume of a sphere by integration.

14. The plane surface of a spherical segment of one base is a circle of radius 8 in., and the greatest thickness of the segment is 4 in. Find its volume by integration.

HINT. What is the radius of the sphere from which the segment was cut?

15. A segment of one base of thickness h is cut from a sphere of radius r. Show by integration that its volume is $\left(\pi\frac{h^2}{3}\right)(3r - h)$.

16. The area bounded by one of the ovals whose equation is $x^2y^2 = (x^2 - 25)(4 - x^2)$ is revolved about the x-axis. Find the volume generated.

17. Find the volume generated by revolving about the x-axis the entire area of the curve whose equation is $y^2 = x^2(4 - x^2)$.

18. Find the volume generated by revolving the area bounded by the loop of the curve whose equation is $y^2 = x(x^2 - 4)$ about the x-axis.

19. The smaller segment of the circle whose equation is $x^2 + y^2 = 25$ cut off by the line whose equation is $x = 3$ is revolved about this line, generating a spindle-shaped solid. Set up the integral giving its volume and evaluate it by Simpson's rule, taking $n = 4$.

20. The area bounded by the parabola whose equation is $y = 4 + 6x - 2x^2$ and the line whose equation is $y = -4$ is revolved about the line. Find the volume generated.

167. Volume of Known Cross Section. Volumes of certain solids may be found by the same general method as is used for volumes of revolution. We shall first develop in general terms the principles used.

The figure represents a solid, whose volume we denote by V, and an axis OX. A set of parallel planes perpendicular to this axis divides the solid into thin plates, one of which is marked in the figure. The volume of any one

of the plates is approximately equal to the product of its thickness Δx and the area of one of its faces, which we denote by $A(x)$, since the cross section of the solid perpendicular to the x-axis is clearly a function of x.

If the volumes of all the plates from $x=a$ to $x=b$ are summed, we get an approximation to the volume of the solid which becomes more exact as $\Delta x \rightarrow 0$; that is,

$$V=\lim \sum A(x)\Delta x.$$

But by the Fundamental Theorem this becomes

$$V=\int_a^b A(x)\,dx.$$

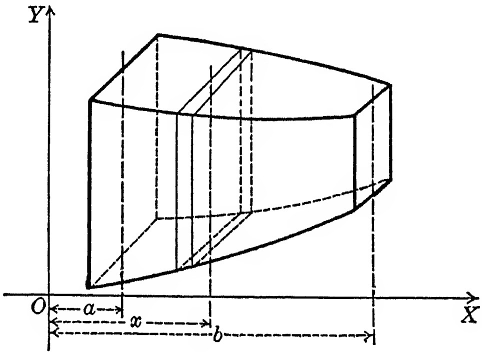


FIG. 163

In applying this formula the solid must be of such a character that the area of any cross section perpendicular to the x-axis (or y-axis) can be represented by a formula in terms of x (or y). The element of volume is a cylinder* whose thickness is dx (or dy) and whose base has the area $A(x)$ (or $A(y)$). Hence

$$dV=A(x)dx \quad \text{or} \quad dV=A(y)dy.$$

We now illustrate the method by examples.

EXAMPLE 1. Derive the formula for the volume of a pyramid.

Solution. Let $OP\,(=h)$ be the altitude of the pyramid $O\text{-}CDE$† and let B denote the area of the base. Let OP be the

*The term *cylinder* is used here in its general sense and includes as special cases circular cylinders and prisms.

†The figure on the following page is drawn with a triangular base, but the reasoning applies to any base.

x-axis with origin at O. Let the segment OP be divided into n parts, which for simplicity will be supposed equal, and let planes parallel to CDE be passed through each point of division, thus dividing the pyramid into n thin plates. Upon each section as a lower base construct a prism with lateral edges parallel to OD and altitude equal to the distance between consecutive sections. The prisms so constructed are called a *set of circumscribed prisms*, and it is obvious that the volume of this set of circumscribed prisms will approach the volume of the pyramid as a limit when n is increased indefinitely.

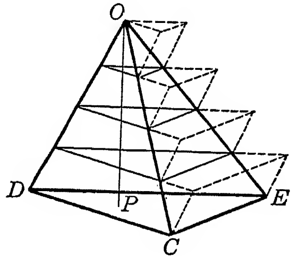


FIG. 164

Let us consider a typical prism, whose volume we denote by dV. The distance of its base from the vertex O is x, and its altitude (or thickness) is dx. If A denotes the area of its base, it is known from solid geometry that

$$\frac{A(x)}{B} = \frac{x^2}{h^2}, \quad \text{or} \quad A(x) = \frac{Bx^2}{h^2}.$$

Hence
$$dV = \frac{Bx^2}{h^2}\,dx.$$

The volume of the set of circumscribed prisms is

$$\sum \frac{Bx^2}{h^2}\,dx,$$

and, by the Fundamental Theorem, the volume of the pyramid is

$$V = \lim \sum \frac{Bx^2}{h^2}\,dx = \int_0^h \frac{Bx^2}{h^2}\,dx = \frac{B}{h^2}\left[\frac{x^3}{3}\right]_0^h = \frac{Bh}{3}.$$

EXAMPLE 2. A solid has a circular base of radius 10 in. The line AB is the diameter of its base, and every section of the solid made by a plane perpendicular to AB is a square. Find the volume.

Solution. In the figure the right-hand end of the solid is cut away to show an element of volume whose base is $PQRS$. Let AB be taken as the x-axis, with the origin at O, the center of the circular base. With this notation the thickness of the element of volume is dx. Since every section perpendicular to AB is a square, the element of volume is a rectangular parallelepiped with square base of edge PQ. In order to express the length PQ in terms of x, we take the y-axis in the plane of the circular base. It is then evident that PQ is a double ordinate of the circle. That is,

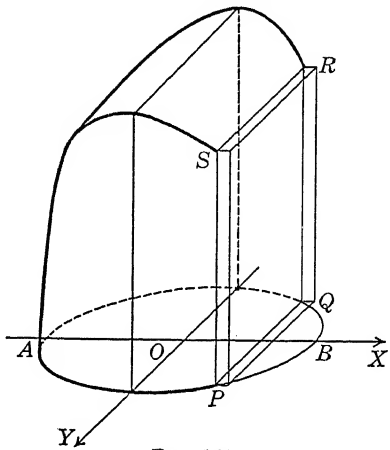


FIG. 165

$$dV = \overline{PQ}^2 dx = 4\,y^2 dx.$$

But the equation of the circle is

$$x^2 + y^2 = 100.$$

Hence

$$dV = 4(100 - x^2)\,dx.$$

The symmetry of the figure shows that half the volume will be obtained by integrating over the segment OB.

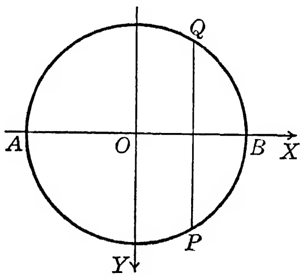


FIG. 166

$$\frac{V}{2} = \int_0^{10} 4(100 - x^2)\,dx = 4\left[100\,x - \frac{x^3}{3}\right]_0^{10} = \frac{8000}{3}.$$

Hence the volume of the solid is

$$V = \tfrac{16000}{3} = 5333.33 \text{ cu. in.}$$

EXAMPLE 3. Find the volume generated by revolving about the y-axis the area bounded by the parabola whose equation is $y = x^2$, the x-axis, and the line whose equation is $x = 2$.

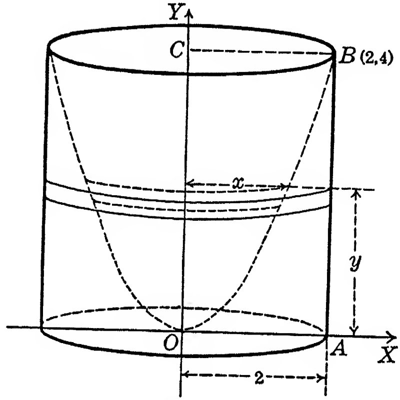


FIG. 167

Solution. If planes are passed through the solid perpendicular to the y-axis, they will cut it into thin plates of thickness dy. The plane surfaces of these plates will be circular rings of which the outer radius is 2 and the inner radius is x. The area of one of these rings will be $4\pi - \pi x^2$.

Hence $$dV = (4\pi - \pi x^2)\,dy$$

and $$V = \pi \int_0^4 (4 - x^2)\,dy.$$

Substituting from the equation of the curve, we have

$$V = \pi \int_0^4 (4 - y)\,dy$$

$$= \pi \left[4y - \frac{y^2}{2}\right]_0^4 = 8\pi.$$

Note that the volume can also be found by subtracting the volumes generated by OBC and $OABC$.

PROBLEMS

1. A solid has a circular base of radius 10 in. The line AB is a diameter of the base. Find the volume of the solid if every section perpendicular to AB is

a. An equilateral triangle.

b. An isosceles right triangle with its hypotenuse in the plane of the base.

c. An isosceles right triangle with one leg in the plane of the base.

**d.* An isosceles triangle with its altitude equal to 10 in.

e. An isosceles triangle with its altitude equal to its base.

2. A solid has a base in the form of an ellipse with major axis 20 in. long and minor axis 10 in. long. Find the volume of the solid if every section perpendicular to the major axis is

a. A square.

b. An equilateral triangle.

**c.* An isosceles triangle with altitude 5 in.

3. The base of a solid is a segment of a parabola cut off by a chord perpendicular to its axis. The chord has a length of 16 in. and is distant 8 in. from the vertex of the parabola. Find the volume of the solid if every section perpendicular to the axis of the base is

a. A square.

b. An equilateral triangle.

c. An isosceles triangle with altitude 10 in.

4. A plane section passed through two opposite seams of a football is an ellipse with major axis 14 in. long and minor axis 7 in. long. Find the volume (*a*) if the leather is so stiff that every section perpendicular to the major axis is a square; (*b*) if every such section is a circle.

5. Two cylinders of equal radius r have their axes meeting at right angles. Find the volume of the common part.

Hint. What is the shape of a section made by a plane parallel to the two axes?

6. The circle whose equation is $x^2 + (y - 6)^2 = 9$ is revolved about the x-axis. The solid generated is called a *torus*. Describe it. Show that the volume is equal to $48\pi\int_0^3 \sqrt{9 - x^2}\,dx$. Evaluate the integral.

*In these problems approximate the definite integrals by Simpson's rule, or use the footnote on page 303.

Since the element of arc on the curve is $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$, the element of surface, dS, may be written

$$dS = 2\,\pi y\,ds.$$

Either of the two forms for ds may be used in applications.

If the arc is revolved about the y-axis, the element of surface is

$$dS = 2\,\pi x\,ds.$$

EXAMPLE. Derive the formula for the area of the surface of a sphere.

Solution. A spherical surface is generated by the revolution of a semicircle about a diameter. Let the x-axis be the axis of revolution and let the semicircle have its center at the origin. The equation of the curve is

$$x^2 + y^2 = r^2;$$

whence $$\frac{dy}{dx} = -\frac{x}{y}.$$

Then

$$ds = \sqrt{1 + \frac{x^2}{y^2}}\,dx = \sqrt{\frac{y^2 + x^2}{y^2}}\,dx = \frac{r}{y}\,dx$$

and $$dS = 2\,\pi y\,ds = 2\,\pi r\,dx.$$

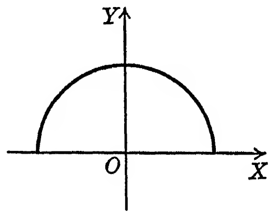


FIG. 170

The limits of integration are from $-r$ to $+r$, or, because of the symmetry, we may integrate from 0 to r and multiply the result by 2.

Hence $$S = 2\int_0^r 2\,\pi r\,dx = 4\,\pi r\left[x\right]_0^r = 4\,\pi r^2.$$

PROBLEMS

1. Find the area of the surface generated by revolving about the y-axis the arc of the parabola $y = x^2$ from $y = 0$ to $y = 2$.

2. Find the area of the surface generated by revolving about the x-axis the arc of the parabola $y^2 = 2\,px$ from $x = 0$ to $x = 4\,p$.

3. Find the area of the surface generated by revolving about the x-axis the arc of $y = x^3$ from $x = 0$ to $x = 2$.

4. Find the area of the surface generated by revolving about the x-axis the arc of the parabola $y^2 = 4 - x$ which lies in the first quadrant.

5. Find the area of the surface generated by revolving about the x-axis the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

*6. Find the area of the surface generated by revolving about the x-axis the arc of the parabola $y = x^2$ from $(0, 0)$ to $(2, 4)$.

*7. Find the area of the surface generated by revolving about the y-axis the arc of $y = x^3$ from $(0, 0)$ to $(2, 8)$.

*8. Find the area of the surface generated by revolving about the x-axis the arc of $y^2 = x^3$ from $(0, 0)$ to $(4, 8)$.

9. The slope of the tractrix at any point of the curve in the first quadrant is given by $\dfrac{dy}{dx} = \dfrac{-y}{\sqrt{c^2 - y^2}}$. Show that the surface generated by revolving about the x-axis the arc joining the points (x_1, y_1) and (x_2, y_2) on the tractrix is $2\pi c(y_1 - y_2)$.

10. The area in the first quadrant bounded by the curves whose equations are $y = x^3$ and $y = 4x$ is revolved about the x-axis. Find the total surface of the solid generated.

11. The area bounded by the y-axis and the curves whose equations are $x^2 = 4y$ and $x - 2y + 4 = 0$ is revolved about the y-axis. Find the total surface of the solid generated.

12. A *zone* is cut from the surface of the sphere generated by revolving the circle whose equation is $x^2 + y^2 = r^2$ about the x-axis by planes perpendicular to the x-axis at $x = a$ and $x = a + h$. Show that the area of the zone is $2\pi rh$.

13. The arc of the curve whose equation is $y^2 = x^3$ extending from the origin to the point $(4, 8)$ is revolved about the y-axis. Find the surface generated.

14. Find the surface generated by revolving about the x-axis the arc of the curve whose equation is $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$ from $x = 1$ to $x = 3$.

* Use Simpson's rule for approximating the definite integral.

15. Find the area of the surface generated by revolving about the x-axis the loop of the curve whose equation is $9y^2 = x(3-x)^2$.

16. Find the area of the surface generated by revolving about the y-axis the loop of the curve whose equation is $9y^2 = x(3-x)^2$.

HINT. Integrate with respect to x.

169. Centroids of Plane Areas. If a piece of stiff cardboard is cut in any shape, there is one point on which it can be balanced. This point is its center of gravity, or, in mathematical terms, it is the centroid of the area represented by the cardboard. Obviously the centroid of a square, of a circle, or of any regular polygon is its geometrical center. In general, if the boundary of any plane area has a center of symmetry, this center will be the centroid of the area. The centroid of a rectangle is its geometrical center, and it has been stated without proof, in Chapter I, that the centroid of a triangular area is the point of intersection of the medians.

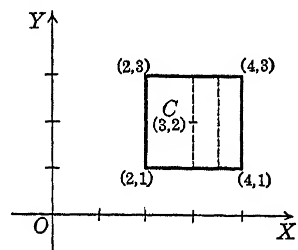


FIG. 171

In order to calculate the position of the centroid of a general plane area, we must first define the terms *moment arm* and *moment of area*. The *moment arm* of an area with respect to an axis is the distance from the axis to the centroid of the area. The *moment of an area* with respect to an axis is the product of an area and its moment arm with respect to that axis.

For example, consider the square whose vertices are (2, 1), (4, 1), (4, 3), (2, 3). The area is 4 sq. in. if the unit of length is 1 in. The centroid is at (3, 2). The moment arm with respect to the y-axis is 3 in., and the moment of area

with respect to the y-axis is $4 \times 3 = 12$.* The moment arm with respect to the x-axis is 2 in., and the moment of area with respect to the x-axis is 8.

Let us now divide the above square into three rectangles by the lines $x = 3$ and $x = 3.5$. The abscissas of the centroids of these rectangles are easily seen to be 2.5, 3.25, and 3.75. Their areas are 2, 1, and 1. Hence the moments with respect to the y-axis are 5, 3.25, and 3.75, and the sum of the moments is 12, which was the moment of the whole square. If the square is divided in any other way, the same result will be obtained. From this example we infer the following principle, which we state without proof:

If an area is divided into any number of parts, the moment of the area about any axis is the sum of the moments of the different parts about the same axis.

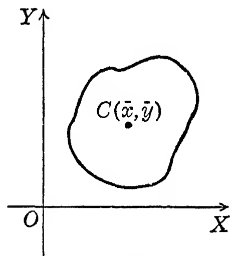


FIG. 172

Now let A denote the number of square units in any given area, M_x the moment of this area with respect to the x-axis, M_y the moment of the area with respect to the y-axis, and $(\bar{x}, \bar{y})$ the coördinates of the centroid of the area. Then, from the definition of moment,

$$M_x = A\bar{y}, \qquad M_y = A\bar{x}.$$

From these equations we get immediately

$$\bar{x} = \frac{M_y}{A}, \qquad \bar{y} = \frac{M_x}{A},$$

from which the coördinates of the centroid may be calculated when the area and its two moments are known.

*Since the moment of area is defined as the product of an area by a length, it is a quantity involving the third power of the unit of length. In engineering applications the unit of length is usually one inch, and the unit of moment of area is written in^3.

In order to calculate M_x and M_y for any area, the area is divided into strips, as in § 163, and the area of each strip is approximated by means of a rectangle. The moment of each element of area may be written down from the figure. The limiting value of the sum of the moments of all the elements is the moment of the whole area, which will then be represented by a definite integral.

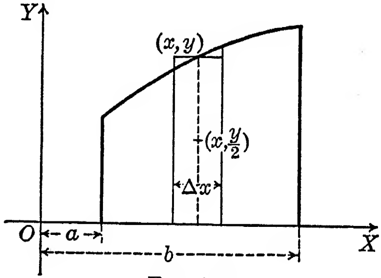


FIG. 173

For the simple case of the area under a curve the process would be as follows: To find M_x and M_y for the area bounded by $y=f(x)$, the x-axis, and the ordinates $x=a$ and $x=b$, the area is divided into rectangular elements by lines parallel to the y-axis. A typical element of width Δx and height y is shown in the figure. Let x be the abscissa of the mid-point of this subinterval Δx. The coördinates of the centroid of this element of area are $\left(x, \dfrac{y}{2}\right)$, and its area is $y\Delta x$. Hence the moment of this element with respect to the x-axis is

$$\frac{y}{2}(y\Delta x)=\frac{1}{2}y^2\Delta x,$$

and, by the Fundamental Theorem,

$$M_x=\int_a^b \tfrac{1}{2}y^2\,dx.$$

Furthermore, the moment of this area with respect to the y-axis is $xy\Delta x$, and

$$M_y=\int_a^b xy\,dx.$$

If not otherwise known, the area is given by

$$A = \int_a^b y\,dx,$$

and the formula on page 341 may be used to calculate the coördinates of the centroid.

In general, if dM_x and dM_y denote the moments of the element dA about the x-axis and the y-axis respectively, and the coördinates of the centroid of dA are (x', y'), then

$$dM_x = y'dA$$

and

$$dM_y = x'dA.$$

It is obvious that an area having an axis of symmetry will have its centroid on this axis. Hence if an area is symmetrical with respect to the x-axis, $\bar{y} = 0$; and if it is symmetrical with respect to the y-axis, $\bar{x} = 0$.

EXAMPLE 1. Find the centroid of the area of a quadrant of a circle.

Solution. Let the axes be chosen so that they bound the quadrant of the circle, as shown in the figure. Then the equation of the circular arc is

$$x^2 + y^2 = r^2.$$

Let the area be divided into elements parallel to the y-axis, one of which is shown. Then, reasoning as above, we have

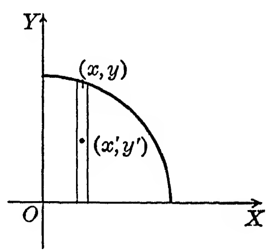


FIG. 174

$$dM_x = \frac{y}{2}dA = \frac{y^2}{2}dx = \frac{1}{2}(r^2 - x^2)\,dx$$

and

$$dM_y = x\,dA = xy\,dx = x\sqrt{r^2 - x^2}\,dx.$$

Hence $M_x = \frac{1}{2}\int_0^r (r^2 - x^2)\,dx = \frac{1}{2}\left[r^2x - \frac{x^3}{3}\right]_0^r = \frac{r^3}{3},$

$$M_y = \int_0^r x\sqrt{r^2 - x^2}\,dx = -\frac{1}{3}\left[(r^2 - x^2)^{\frac{3}{2}}\right]_0^r = \frac{r^3}{3}.$$

The area of a quadrant of a circle is $\frac{\pi r^2}{4}$. Hence we have

$$\bar{x} = \frac{r^3}{3} \div \frac{\pi r^2}{4} = \frac{4r}{3\pi}, \qquad \bar{y} = \frac{r^3}{3} \div \frac{\pi r^2}{4} = \frac{4r}{3\pi}.$$

EXAMPLE 2. Find the centroid of the area bounded by the curves whose equations are $y = x^2$ and $y = x + 2$.

Solution. Inspection of the figure shows that the simplest way to take the elementary strips is parallel to the y-axis. Let x denote the abscissa of the mid-point of the typical subinterval Δx, and y_1 and y_2 denote the ordinates of the corresponding points on the line and the parabola respectively. Then the area of the element is $(y_1 - y_2)\Delta x$, and the centroid of the element will have the coördinates

$$[x, \tfrac{1}{2}(y_1 + y_2)].$$

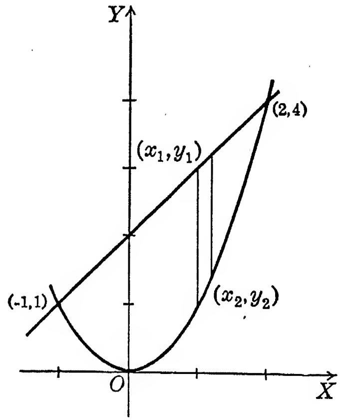


FIG. 175

Hence $\qquad dA = (y_1 - y_2)\,dx = (x + 2 - x^2)\,dx;$

$$dM_x = \frac{y_1 + y_2}{2}\,dA$$

$$= \frac{y_1^2 - y_2^2}{2}\,dx$$

$$= \frac{x^2 + 4x + 4 - x^4}{2}\,dx;$$

$$dM_y = x\,dA$$

$$= x(x + 2 - x^2)\,dx$$

$$= (x^2 + 2x - x^3)\,dx.$$

Therefore

$$A=\int_{-1}^{2}(x+2-x^2)\,dx=\left[\frac{x^2}{2}+2x-\frac{x^3}{3}\right]_{-1}^{2}=\frac{9}{2};$$

$$M_x=\frac{1}{2}\int_{-1}^{2}(x^2+4x+4-x^4)\,dx=\frac{1}{2}\left[\frac{x^3}{3}+2x^2+4x-\frac{x^5}{5}\right]_{-1}^{2}=\frac{36}{5};$$

$$M_y=\int_{-1}^{2}(x^2+2x-x^3)\,dx=\left[\frac{x^3}{3}+x^2-\frac{x^4}{4}\right]_{-1}^{2}=\frac{9}{4}.$$

Hence the coördinates of the centroid are

$$\bar{x}=\tfrac{9}{4}\div\tfrac{9}{2}=\tfrac{1}{2},\quad \bar{y}=\tfrac{36}{5}\div\tfrac{9}{2}=\tfrac{8}{5}.$$

PROBLEMS

Find the centroids of the areas bounded by the following curves:

1. $y^2=x,\ x=4.$
2. $y^2=2px,\ x=h.$
3. $y^2=x,\ y=2,\ x=0.$
4. $y=x^3,\ x=2,\ y=0.$
5. $y=x^3,\ y=8,\ x=0.$
6. $y^2=x^3,\ x=4,\ y=0.$
7. $y^2=x^3,\ y=8,\ x=0.$
8. $y=x^3,\ y=4x.$
9. $y=6x-x^2,\ y=x.$
10. $x=4y-y^2,\ y=x.$
11. $y=4x-x^2,\ y=2x-3.$
12. $y^2=4x,\ 2x-y=4.$
13. $y=x^3-3x,\ y=x.$
14. $y=x^2,\ 2x-y+3=0.$
15. $y=x^2-2x-3,$
    $y=6x-x^2-3.$

16. Find the centroid of the area of the triangle whose vertices are $(0, 0)$, $(a, 0)$, and $(0, b)$.

17. Find the centroid of the area bounded by the parabola whose equation is $\sqrt{x}+\sqrt{y}=\sqrt{a}$ and the coördinate axes.

18. Find the centroid of the area bounded by the loop of the curve whose equation is $y^2=4x^2-x^3$.

19. Find the centroid of the portion in the first quadrant of the ellipse whose equation is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

HINT. The area of the ellipse is πab.

170. Centroids of Other Figures. The definitions and methods of the previous section may be easily extended to the finding of centroids of volumes, surfaces of revolution, and arcs of curves. The chief difference is that the moment of volume with respect to an axis is the product of the *volume* by the moment arm, etc. A thorough consideration of the subject belongs to more advanced courses, but it may be of interest to work a few simple problems.

Suppose, for example, that we wish to find the centroid of the solid generated by revolving about the x-axis the segment of the parabola $y^2 = 2px$ cut off by the line $x = h$.

The centroid is on the x-axis. Cutting the solid by planes perpendicular to the x-axis, we have elements of volume

$$dV = \pi y^2 dx.$$

The moment of each of these with respect to the y-axis is

$$dM_y = x\,dV = \pi x y^2 dx.$$

Hence the moment of the solid is

$$M_y = \pi \int_0^h xy^2\,dx = \pi \int_0^h 2\,px^2\,dx = \frac{2\,\pi p h^3}{3}.$$

On the other hand, the volume of the solid is

$$V = \pi \int_0^h y^2\,dx = \pi p h^2.$$

Hence $$\bar{x} = M_y \div V = \frac{2\,h}{3}.$$

PROBLEMS

1. Find the distance of the centroid of a solid hemisphere of radius r from the center.

2. Find the distance of the centroid of the arc of a semicircle of radius r from the center.

3. Find the distance of the centroid of the area of the curved surface of a hemisphere of radius r from the center.

171. Fluid Pressure. Whenever water (or another fluid) presses against a vertical retaining surface, the pressure on the surface is different at different depths below the surface of the water. In order to determine the pressure at a given depth, we make use of the physical principle that pressure at a point within a fluid is the same in all directions. Hence the pressure against a vertical surface at a given point is equal to the pressure downward at that point, and the pressure will be the same at all points which are the same distance below the surface. The method of applying these general principles will be illustrated by some examples.

EXAMPLE 1. One side of a rectangular water tank is 10 ft. long and 4 ft. deep. Calculate the pressure on this side when the tank is full of water.

Solution. Let the vertical edge OA of the side of the tank be taken as the x-axis, and let the segment OA be divided into n parts. Horizontal lines through the points of division will divide the side of the tank into elementary strips. Let Δx be the width of any one of these strips and let x be the depth to the center of this strip. Imagine this vertical strip to be turned about its center line until its surface is horizontal. The pressure on it would then be equal to the weight of water supported by it. The volume of water supported by it is equal to its area, $10\,\Delta x$, times the depth, x. If w denotes the weight of a cubic foot of water, the weight on this horizontal strip would be $w\,10\,x\,\Delta x$.

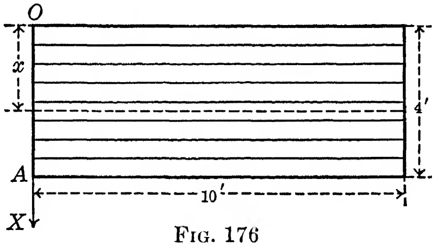


FIG. 176

Now, since the pressure is the same in all directions and since the strip is very narrow, the pressure is approximately the same on the strip in its vertical position as in its horizontal

position. Hence the pressure above may be taken as representing approximately the pressure on the typical strip, and an approximation to the value of the pressure on the whole side will be given by

$$\sum w\, 10x\, \Delta x.$$

As the width of each strip approaches zero as a limit this approximate value approaches the exact value of the pressure as a limit. Hence, by the Fundamental Theorem, the pressure P on the side of the tank is given by

$$P = \lim \sum w\, 10x\, \Delta x = \int_0^4 w\, 10x\, dx = 10\, w \left[\frac{x^2}{2}\right]_0^4 = 80\, w.$$

The weight of a cubic foot of water is about 62.5 lb. Using this value for w gives as the final result 5000 lb.

The essential part of the above reasoning is that the pressure on an elementary horizontal strip is equal to the product of the area of the strip (dA), the depth of the strip (h), and the weight of a cubic unit of the fluid (w). That is,

$$\boldsymbol{dP = wh\, dA.}$$

Example 2. The vertical end of a water trough is in the form of a right triangle with dimensions as shown in the figure. Calculate the pressure on this end when the trough is full of water.

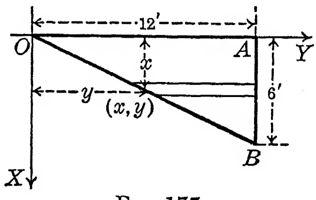


Fig. 177

Solution. Imagine that the triangular area is divided into elementary strips, one of which is shown in the figure, and that coördinate axes are introduced as shown. The equation of the line OB, which joins the origin to the point B (6, 12), is $y = 2x$. With (x, y) representing the coördinates of a point on OB, it may be seen from the figure that

1. The width of the elementary strip is dx and its length is $12 - y$. Hence $dA = (12 - y)\, dx$.

2. The depth of the elementary strip is x.

3. The limits of integration are from $x = 0$ to $x = 6$. Hence, reasoning as in Example 1, the element of pressure dP is

$$dP = wx(12 - y)\,dx = 62.5\,x(12 - 2\,x)\,dx.$$

Hence the total pressure is

$$P = 62.5\int_0^6 x(12 - 2\,x)\,dx = 62.5\left[6\,x^2 - \tfrac{2}{3}\,x^3\right]_0^6 = 4500 \text{ lb.}$$

NOTE. The purpose of the particular choice of axes in the above example was to avoid negative coördinates. This is usually the best way to draw the axes in pressure problems if the origin is taken at the top of the surface; if the origin is taken at the bottom (as in the next example), the axes may be conveniently drawn in the usual way.

EXAMPLE 3. A gate in a dam is in the form of an isosceles trapezoid, as shown in the figure. Calculate the pressure on the gate when the surface of the water is 8 ft. above the top of the gate.

Solution. Choosing rectangular axes as shown, we see from the figure that

1. $dA = 2\,x\,dy.$

2. The depth of the elementary strip below the surface is $12 - y$.

3. The limits of integration are $y = 0$ to $y = 4$.

4. The equation of AB is $y = 2\,x - 8$. Hence

$$dP = 62.5(12 - y)2\,x\,dy$$
$$= 62.5(y + 8)(12 - y)\,dy,$$

and the total pressure is

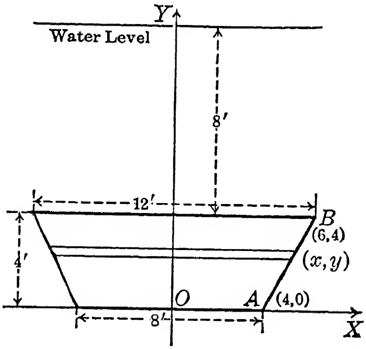


FIG. 178

$$P = 62.5\int_0^4 (y + 8)(12 - y)\,dy$$
$$= 62.5\int_0^4 (96 + 4\,y - y^2)\,dy$$
$$= 62.5\left[96\,y + 2\,y^2 - \frac{y^3}{3}\right]_0^4 = 24{,}667 \text{ lb.}$$

PROBLEMS

1. A horizontal cylindrical tank of diameter 8 ft. is half full of oil weighing 60 lb./cu. ft. Calculate the pressure on one end.

2. Each end of a horizontal oil tank is an ellipse of which the horizontal axis is 12 ft. long and the vertical axis 6 ft. long. Calculate the pressure on one end when the tank is half full of oil weighing 60 lb./cu. ft.

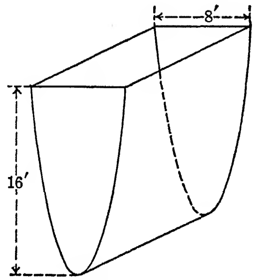


FIG. 179

3. The vertical end of a vat is a segment of a parabola 8 ft. across the top and 16 ft. deep (Fig. 179). Calculate the pressure on this end when the vat is full of a liquid weighing 70 lb./cu. ft.

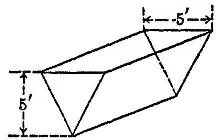


FIG. 180

4. The vertical end of a water trough is an isosceles triangle with dimensions as shown in Fig. 180. Calculate the pressure on the end when the trough is full of water.

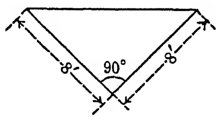


FIG. 181

5. The vertical end of a water trough is an isosceles right triangle of which each leg is 8 ft. (Fig. 181). Calculate the pressure on the end when the trough is full of water.

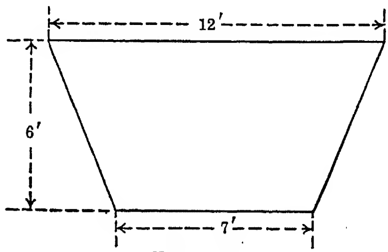


FIG. 182

6. The vertical end of a trough is an isosceles trapezoid with dimensions as shown in Fig. 182. Calculate the pressure on the end when the trough is filled with a liquid weighing 50 lb./cu. ft.

7. The vertical end of a tank is an isosceles trapezoid with dimensions as shown in Fig. 183. Calculate the pressure on the end when the tank is filled with a liquid weighing 50 lb./cu. ft.

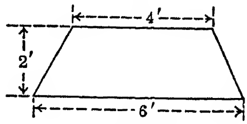


FIG. 183

8. A gate in a dam is in the form of a right triangle with dimensions as shown in Fig. 184. Calculate the pressure on the gate when the surface of the water is 6 ft. above the top of the gate.

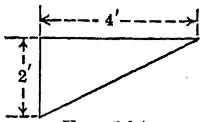


FIG. 184

9. A gate in a dam is in the form of a right triangle as shown in Fig. 185. Calculate the pressure on the gate when the surface of the water is 4 ft. above the top of the gate.

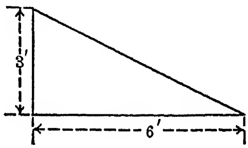


FIG. 185

10. A gate in a dam is in the form of an isosceles trapezoid as shown in Fig. 186. Calculate the pressure on the gate when the surface of the water is 8 ft. above the top of the gate.

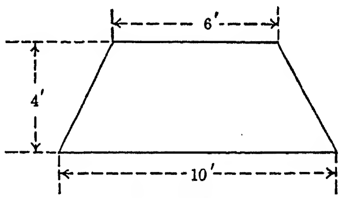


FIG. 186

11. A rectangular gate in a vertical dam is 10 ft. wide and 6 ft. deep. Find (*a*) the pressure when the level of the water is 8 ft. above the top of the gate; (*b*) how much higher the water must rise to double the pressure found in (*a*).

12. A vertical dam has the shape of a parabola; it is 600 ft. across the top and 40 ft. deep in the center. Find the pressure upon it when the water is 30 ft. deep.

13. Show that the pressure on any vertical surface is the product of the weight of a cubic unit of the liquid, the area of the surface, and the depth of the centroid.

14. A vertical cylindrical tank, of diameter 30 ft. and height 50 ft., is full of water. Find the pressure on the curved surface.

172. Work. *Work* is a technical term used in mechanics. The work done in overcoming a resistance is defined as the product of the distance through which a body is moved and of the force opposed to the motion. In the applications of this section the only force considered is the force of gravity, and the work considered is that done in lifting a body. In this case the work done is the product of the weight of the body by the vertical distance through which it is lifted. The usual unit of work is the foot-pound (ft.-lb.). Thus, if a weight of 100 lb. is lifted upward 2 ft., the amount of work done is 200 ft.-lb.

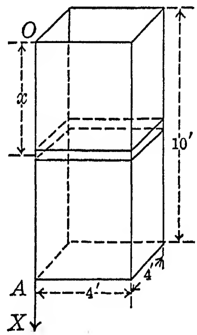


Fig. 187

The use of the calculus in calculating work is illustrated by the following simple example. Suppose that a rectangular tank 10 ft. deep, with a base 4 ft. square, is filled with water. How much work must be done to pump the water to the top of the tank? The definition given above does not apply immediately, because the water near the bottom of the tank must be lifted much higher than the water near the top. To answer the question, we must apply the Fundamental Theorem of the Integral Calculus.

Let the edge OA of the tank be taken as the x-axis and let the segment OA be divided into n parts. Imagine horizontal planes passed through each point of division so that consecutive planes divide the water into horizontal layers. Let us fix our attention on a typical layer of thickness Δx and let x be the depth to some point in this layer. Assuming that Δx is small, each drop of water in the layer must be lifted a distance nearly equal to x, and hence an approximate value of the work done in lifting this layer to the top will be x times the weight of the water in the layer.

If w denotes the weight of a cubic foot of water, the weight of the layer is $w\,16\,\Delta x$, and an approximate value of the work done is $w\,16\,x\Delta x$.

An approximation to the total work done will be

$$\sum w 16 x\Delta x,$$

the summation extending over all the layers from $x = 0$ to $x = 10$.

As the thickness of each layer approaches zero as a limit this approximate value approaches the exact value of the work as a limit. Hence, by the Fundamental Theorem, the work W is given by

$$W = \lim \sum w 16\, x\Delta x = \int_0^{10} 16\, wx\,dx$$
$$= 16\,(62.5)\left[\frac{x^2}{2}\right]_0^{10} = 50{,}000 \text{ ft.-lb.}$$

The essential principle of the reasoning in this example is that the element of work (dW) done in lifting an elementary layer of water (dV) is equal to the weight of the layer multiplied by the height (h) it is lifted.

That is, $$dW = hwdV.$$

EXAMPLE. A conical cistern is 20 ft. across the top and 15 ft. deep. If the surface of the water is 5 ft. from the top, calculate the work necessary to pump the water to the top of the cistern.

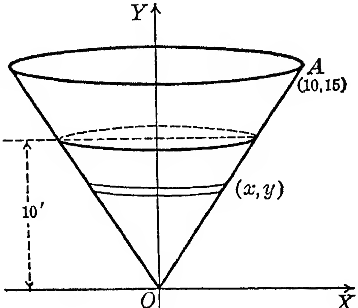


FIG. 188

Solution. Let a system of rectangular axes be introduced with the origin at the vertex of the cone and with the x-axis horizontal. The element of the cone lying in the xy-plane joins (0, 0) and

(10, 15); hence its equation is $2y = 3x$. An elementary horizontal layer of water is shown in the figure. If (x, y) is a point on OA, the following facts appear from the figure:

1. The thickness of the layer of water is dy.
2. The radius of the layer of water is x.
3. The height which this layer must be raised is $15 - y$.
4. The limits of integration are from $y = 0$ to $y = 10$.

The volume of the elementary layer is

$$dV = \pi x^2 dy = \tfrac{4}{9}\pi y^2 dy.$$

Hence
$$\begin{aligned} dW &= (15 - y) w\, dV \\ &= (15 - y)(62.5)(\tfrac{4}{9}\pi y^2 dy) \end{aligned}$$

and
$$\begin{aligned} W &= \frac{4\pi}{9}(62.5)\int_0^{10}(15y^2 - y^3)\,dy \\ &= \frac{4\pi}{9}(62.5)\left[5y^3 - \frac{y^4}{4}\right]_0^{10} = \frac{625000\,\pi}{9} \\ &= 218{,}167 \text{ ft.-lb.} \end{aligned}$$

PROBLEMS

1. A vertical cylindrical cistern of diameter 16 ft. and depth 20 ft. is full of water. Calculate the work necessary to pump the water to the top of the cistern.

2. A vertical cylindrical cistern of diameter 10 ft. and depth 12 ft. is half full of water. Calculate the work necessary to pump the water to the top of the cistern.

3. A vertical cylindrical cistern of diameter 12 ft. and depth 12 ft. is full of water. Calculate the work necessary to pump the water to a height of 10 ft. above the top of the cistern.

4. A conical cistern 20 ft. across the top and 20 ft. deep is full of water. Calculate the work necessary to pump the water to a height of 15 ft. above the top of the cistern.

5. A hemispherical tank of diameter 10 ft. is full of oil weighing 60 lb./cu. ft. Calculate the work necessary to pump the oil to the top of the tank.

6. A vat is built in the shape of a regular pyramid 12 ft. square at the top and 10 ft. deep. If the vat is full of a liquid weighing 65 lb./cu. ft., calculate the work necessary to pump the liquid to the top of the vat.

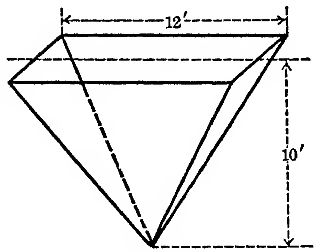


FIG. 189

7. Calculate the work necessary to pump the liquid in the vat of Problem 6 to a height of 8 ft. above the top of the vat.

8. The ends of a horizontal trough are vertical isosceles right triangles. The width across the top is 8 ft., and the length is 10 ft. If the trough is full of water, calculate the work necessary to pump the water to the top of the trough.

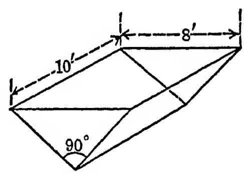


FIG. 190

9. Each end of a trough is a vertical right triangle. The dimensions are shown in Fig. 191. If the trough is full of water, calculate the work necessary to pump the water to the top of the trough.

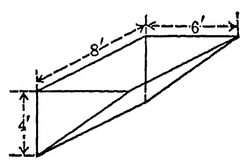


FIG. 191

10. Each end of a trough is a vertical isosceles trapezoid. The dimensions are shown in Fig. 192. If the trough is full of water, calculate the work necessary to pump the water to the top of the trough.

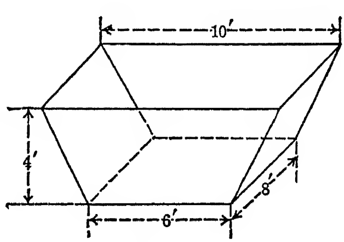


FIG. 192

11. A bucket of weight M is to be lifted from the bottom of a shaft h feet deep. The weight of the rope used to hoist it is m pounds per foot. Find the work done.

CHAPTER XIV

THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS

173. The functions so far used in this book have been algebraic; that is, combinations of the variables and constants involving sums, products, quotients, and constant powers and roots. All other functions are called *transcendental.* The more common transcendental functions are the exponential, logarithmic, trigonometric, and inverse trigonometric functions.

174. The Exponential Function. The quantity $y = a^x$, in which a is a constant, is known as an *exponential function.* The number a is called the base. In elementary work it is always positive and usually greater than 1.

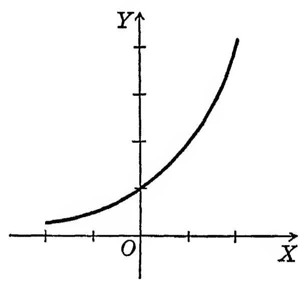


FIG. 193

In the following discussion of the properties of the exponential function it will be assumed always that $a > 1$. When $x = 0$, $y = a^0 = 1$. Since a is positive, a^x is positive for all values of x except fractional ones with even denominators. To avoid exceptions, we always use the positive sign when two signs are possible; thus, when $x = \frac{3}{2}$, we consider $a^x = a^{\frac{3}{2}} = +\sqrt{a^3}$. Hence the graph of $y = a^x$ crosses the y-axis at (0, 1) and is everywhere above the x-axis.

Obviously the graph has neither an axis nor a center of symmetry.

Since $a > 1$, a^x increases indefinitely as $x \rightarrow +\infty$. Hence the curve recedes indefinitely from both axes in the first quadrant. When x is negative, a^x is less than 1; and as $x \rightarrow -\infty$, $a^x \rightarrow 0$.* Hence the curve approaches the x-axis as an asymptote in the second quadrant.

The computation of the table of values and the plotting are performed in the usual way. The above graph is that for $a = 2$.

175. The Number *e*. For reasons which will appear later, the most important exponential function is that for which the base is equal to a certain irrational number whose value is approximately 2.71828. This number is denoted by the letter e, and its value should be memorized. The function $y = e^x$ is usually called the exponential function.

The number e is just as necessary in the calculus as the number $\pi (= 3.14159)$ in geometry. It is defined as $\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n$. In higher mathematics it is proved that this limit exists, that it is irrational, and that it is approximately equal to 2.71828. This approximation may be verified by actual multiplication when n is small, or by the aid of a table of logarithms when n is large. Thus $\left(1 + \frac{1}{n}\right)^n = 2.25$ when $n = 2$; $\left(1 + \frac{1}{n}\right)^n = 2.370$ when $n = 3$; $\left(1 + \frac{1}{n}\right)^n = 2.705$ when $n = 100$; etc.

Since the discussion in § 174 is general for $a > 1$, the graph of $y = e^x$ will have the properties there enumerated for $y = a^x$ and will closely resemble that of $y = 2^x$. But it will recede more rapidly from the x-axis in the first quadrant

* This is readily seen by setting $x = -u$. Then $a^x = a^{-u} = \frac{1}{a^u}$. Now as $x \rightarrow -\infty$, $u \rightarrow +\infty$ and $a^u \rightarrow \infty$; therefore $a^x \rightarrow 0$.

and will be closer to the x-axis in the second quadrant, as is indicated by the following table of values:

x	-4	-3	-2	-1	0	1	2	3	4
2^x	0.0625	0.1250	0.250	0.500	1.0	2.000	4.000	8.00	16.00
e^x	0.0183	0.0498	0.135	0.368	1.0	2.718	7.389	20.09	54.60

The values of e^x may be calculated by substituting $e = 2.71828$ and using logarithms. Since the function e^x is so important, however, its values are given for numerous values of x in any good set of tables.

PROBLEMS

1. Plot the graphs of the following functions:

a. $y = 3^x$. *d.* $y = 10^{-x}$. *g.* $y = 1^x$.
b. $y = (\frac{1}{2})^x$. *e.* $y = (\frac{1}{4})^x$. *h.* $y = 2^{-x^2}$.
c. $y = e^{-x}$. *f.* $y = 3^{2x}$.

2. Show that the definition of e given above is equivalent to

$$e = \lim_{k \to 0}(1 + k)^{\frac{1}{k}}.$$

3. Find from the tables the values of the following functions:

a. $e^{3.16}$. *b.* $e^{-2.45}$. *c.* $e^{0.017}$. *d.* $e^{-0.0764}$.

4. Calculate the values of the powers of e in Problem 3 by means of logarithms.

5. Discuss $y = a^x$ for the case in which a is less than one.

6. What happens to the graph of $y = a^x$ when a becomes very large? when a approaches 0?

7. Draw the graphs of the following functions roughly by inspection, showing merely the general shape and location:

a. $y = 10\, e^{0.4x}$. *c.* $y = 0.06\, e^{5x}$.
b. $y = 1000\, e^{-4x}$. *d.* $y = 0.2\, e^{-0.002t}$.

8. Sketch the following curves on the same axes:

a. $y = e^x$. *b.* $y = e^{-x}$. *c.* $y = -e^x$. *d.* $y = -e^{-x}$.

9. Plot the graph of $y = e^{-x^2}$ (probability curve).

10. Plot the graph of $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ (catenary).

11. Plot the graph of $y = \frac{1}{2}(e^x - e^{-x})$ (hyperbolic sine curve; the catenary for $a = 1$ is called the hyperbolic cosine).

12. Plot the graph of each of the following functions:

a. $y = e^{x-4}$.

b. $y = e^{5-x}$.

c. $y = e^{\frac{1}{x}}$.

d. $y = e^{\sqrt{x}}$.

e. $y = e^x - x$.

f. $y = \dfrac{e^x}{x}$.

g. $y = \dfrac{xe^x}{10}$.

h. $y = \dfrac{xe^{-x^2}}{10}$.

176. The Logarithmic Function. Since $y = a^x$ is a function of x, x is also a function of y. This inverse function is the logarithm of y to the base a by virtue of the definition of logarithms, which is as follows:

The logarithm of a number to any base is the exponent of the power to which the base must be raised to equal the number.

In other words, if y denotes the number, a denotes the base, and x denotes the logarithm of y to the base a, the above definition may be stated algebraically as follows:

$$\boldsymbol{x = \log_a y \text{ if and only if } y = a^x.}$$

The equivalence of the two equations $y = a^x$ and $x = \log_a y$ is perhaps the most important single fact in connection with logarithms.

Example 1. Find the value of $\log_9 27$.

Solution. Here the number is 27 and the base is 9. Using the notation of the definition above, we have $y = 27$ and $a = 9$. Hence we are required to find the power of 9 which equals

27, or to find the value of x satisfying the equation $27 = 9^x$. This equation is easily solved, since both 9 and 27 are powers of 3. It is equivalent to

$$3^3 = 3^{2x};$$

whence $$2x = 3, \quad x = \tfrac{3}{2}, \quad \text{or} \quad \log_9 27 = \tfrac{3}{2}.$$

EXAMPLE 2. What is the number whose logarithm to the base 2 is -3?

Solution. In this case $a = 2$ and $x = \log_2 y = -3$. Therefore

$$y = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}.$$

The properties of logarithms are readily deduced from those of the exponential function given in § 174. The base a must be positive and different from 1. Since $a^0 = 1$, $\log_a 1 = 0$. Since $y = a^x$ is positive for all real values of x, $\log_a y$ has no meaning for y negative.* Taking a greater than 1, as is usually the case, the discussion of § 174 shows that logarithms of numbers greater than 1 are positive and that those of numbers less than 1 are negative, since $y = a^x$ is greater than 1 for x positive and less than 1 for x negative. Also, when y increases indefinitely, $x = \log_a y$ approaches $+\infty$; and when $y \rightarrow 0$, x approaches $-\infty$. Note also that since y is never 0, 0 has no logarithm. These facts are best memorized by memorizing the graph of $y = a^x$ given in § 174.

177. Graph of the Logarithmic Function. Interchanging variables, so as to make y the dependent variable, we have $y = \log_a x$. This, by definition, is equivalent to $x = a^y$. Hence the graph of $y = \log_a x$ is the same as that of $y = a^x$ with the axes interchanged.

*In higher mathematics it is shown that negative numbers have imaginary logarithms. These will not be considered in this course.

Or, referring to the latter part of the previous section, we may proceed as follows. Since $\log_a 1 = 0$, the graph cuts the x-axis at (1, 0); since 0 has no logarithm, the graph does not cut the y-axis. For $x > 1$, $y = \log_a x$ is positive and increases indefinitely with x. For $x < 1$, $y = \log_a x$ is negative and approaches $-\infty$ as $x \rightarrow 0$. For x negative, y has no real value. Hence the graph is confined to the first and fourth quadrants; it recedes indefinitely from both axes in the first quadrant and approaches the y-axis as an asymptote in the fourth. The figure shows the graph of $y = \log_2 x$.

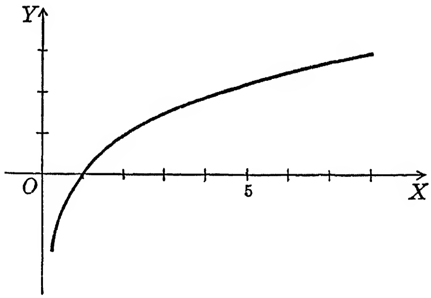


FIG. 194

178. The Two Bases. Any positive number, except 1, can be used as the base of a system of logarithms, but for practical purposes we are restricted to two bases.

The logarithms used in computation have the base 10; these are the logarithms used by the student in his previous work. The reason for the choice of 10 as the base of the logarithms of computation work is that our number system is decimal. A shifting of the decimal point merely multiplies (or divides) the number by 10, 100, 1000, etc. The effect on the logarithm is, therefore, to add to it (or subtract from it) 1, 2, 3, etc., since $\log 10 = 1$, $\log 100 = 2$, $\log 1000 = 3$, etc. In other words, only the characteristic is affected, and therefore the same tables of mantissas may be used for numbers 1–99, 10–990, 100–9900, etc. Logarithms to the base 10 are called *common*, or *Briggs*, logarithms.

It is customary in calculation with common logarithms to keep the mantissa always positive by the device of adding and subtracting 10, or by writing a minus sign over the characteristic. Thus we write

$$\log_{10} 0.5 = 9.6990 - 10,$$

or $$\log_{10} 0.5 = \bar{1}.6990 \text{ (which means } 0.6990 - 1).$$

The actual value of $\log_{10} 0.5$ is -0.3010, and this is the value which must be used in plotting.

The base of the logarithms used most in calculus is the irrational number $e = 2.71828 \cdots$. Logarithms to this base are called *natural*, or *Napierian*, logarithms. Since they are the usual ones, it will henceforth be understood that when no base is stated the base is e. That is, $\log x$ means $\log_e x$.

It is assumed that the student is familiar with the use of tables of common logarithms. In the better tables Napierian logarithms are also given for a short range of values of x. The method of finding $\log x$ when x is without this range of values is best explained by an example. Suppose that it is desired to find log 446. We first write 446 as a product of a power of 10 and a number comprised in our table, thus:

$$446 = 4.46 \times 10^2.$$

Hence $$\log 446 = \log 4.46 + 2 \log 10.$$

From the tables, $$\log 10 = 2.30259.$$

$$\begin{array}{r} 2 \log 10 = 4.605 \\ \underline{\log 4.46 = 1.495} \\ \log 446 = 6.100 \end{array}$$

Adding,

A rough check is afforded by the tables of powers of e, which give $e^6 = 403.4$.

Similarly, $$\log 0.0068 = \log\left(\frac{6.8}{1000}\right)$$
$$= \log 6.8 - 3\log 10$$
$$= 1.917 - 6.908$$
$$= -4.991.$$

Check. $e^{-5} = 0.00674.$

179. Theorems on Logarithms. The following theorems are usually given in advanced algebras and trigonometries. They are reproduced here as formulas for convenience of reference.

$$\text{(I)} \qquad \log_a xy = \log_a x + \log_a y.$$

$$\text{(II)} \qquad \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y.$$

$$\text{(III)} \qquad \log_a x^n = n \log_a x.$$

$$\text{(IV)} \qquad \log_a \sqrt[n]{x} = \frac{1}{n}\log_a x.$$

$$\text{(V)} \qquad \log_b x = \frac{\log_a x}{\log_a b}.$$

These formulas are all proved by reference to the definition of logarithms. As Formula V is not commonly used in elementary work, the proof will be given here.

By definition, if we set $u = \log_b x$, $b^u = x$.

Taking the logarithms of both sides by Formula III,

$$u \log_a b = \log_a x.$$

Hence $$u = \frac{\log_a x}{\log_a b},$$

or $$\log_b x = \frac{\log_a x}{\log_a b}.$$

This last formula is used for transferring logarithms from one base to another. For example, the Napierian logarithm

of 446 may be found as follows from a table of common logarithms:

$$\log_{10} 446 = 2.6493$$

and

$$\log_{10} e = \log_{10} 2.718 = 0.4343.$$

Hence

$$\log_e 446 = \frac{2.6493}{0.4343} = 6.1002.$$

If in Formula V we set $x = a$, we get

$$\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b}.$$

This relation enables us to write Formula V in the form

$$\textbf{(Va)} \qquad \log_b x = \log_a x \log_b a.$$

When $b = 10$ and $a = e$, the quantity $\log_{10} e$ is known as the *modulus* of the common system of logarithms; its value is 0.43429, the reciprocal of $\log_e 10 = 2.30259$.

EXAMPLE. Discuss and sketch the graph of $y = \log \sqrt{25 - x^2}$.

Solution. We first simplify the equation by means of Formula IV, obtaining

$$y = \tfrac{1}{2} \log (25 - x^2).$$

1. *Intercepts.* If $x = 0$, then $y = \frac{1}{2} \log 25 = \log 5 = 1.6$. If $y = 0$, then $25 - x^2 = 1$ (since $\log 1 = 0$) and $x = \pm\sqrt{24} = \pm 4.9$.

2. *Symmetry.* The graph is symmetrical with respect to the y-axis only.

3. *Extent.* Since negative numbers have no real logarithms, x^2 cannot exceed 25. Hence the graph does not extend to the right of $x = 5$ nor to the left of $x = -5$.

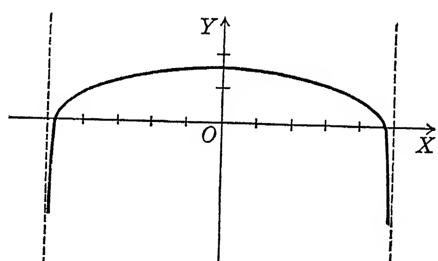


FIG. 195

4. *Asymptotes.* Since $y \to -\infty$ when $(25 - x^2) \to 0$, the graph has two vertical asymptotes, $x = \pm 5$.

PROBLEMS

1. Find the value of x in each of the following equations:

a. $x = \log_2 32$. *c.* $\log_x \frac{1}{8} = -2$.
b. $x = \log_{100} 1000$. *d.* $\log_5 x = -3$.

2. Using the definition of logarithms, solve for x, in terms of y, the equations in Problem 1 on page 358.

3. Write in exponential form

a. $y = \log_3 x$. *c.* $y + \log_{10} x = 0$.
b. $2y = \log 6x$. *d.* $\log y^2 = -6x$.

4. Prove that if $y = x^{\frac{1}{\log x}}$, then $y = e$.

5. Look up the natural logarithms of 6.87, 0.00276, 786.

6. Look up the numbers whose natural logarithms are 0.2673, 6.7200, − 0.0276.

7. Express as powers of 10

a. $e^{0.02t}$. *b.* $e^{2.6}$. *c.* $e^{-0.4t}$.

Hint. What power of 10 is e?

8. Express as powers of e

a. 10^{-x}. *b.* 2^{-3x}. *c.* $10^{0.4x}$.

Hint. What power of e is 10?

9. Calculate the following:

a. $\log 7^{3.2}$. *c.* $5.4^{6.2}$. *e.* $0.45^{2.4}$.
b. $\log 0.3^{0.6}$. *d.* $2^{4.53}$. *f.* $10^{1.2} + 10^{-1.2}$.

10. Solve the following equations for x in terms of y:

a. $y = \frac{1}{2}(e^x + e^{-x})$. *b.* $y = \frac{1}{2}(e^x - e^{-x})$.

11. Sketch the graphs of $y = \log_e x$ and $y = \log_{10} x$ on the same axes.

12. Sketch the graphs of $y = \log x$, $y = \log \dfrac{1}{x}$, $y = \log(-x)$, and $y = \log\left(-\dfrac{1}{x}\right)$ on the same axes.

13. Sketch the graphs of $y = \log x$, $y = \log x^3$, and $y = \log \sqrt{x}$ on the same axes.

14. Sketch the graphs of $y = \log x$, $y = \log 10x$, and $y = \log \frac{x}{10}$ on the same axes.

15. Discuss and sketch the graphs of the following equations:

a. $y = \log (2 + x)$.
b. $y = \log (2 - x)$.
c. $y = \log \sqrt{x - 4}$.
d. $y = \log (1 - x^2)$.
e. $y = \log (10 - 2x)$.
f. $y = \log \left(\frac{1}{3 + x}\right)$.
g. $y = \log \left(\frac{1}{25 - x^2}\right)$.
h. $y = \log (16 - x^2)^3$.

16. Discuss and plot the graph of $y = \log (36 - x^2)$. Using this graph and formulas II and IV, sketch the graphs of $y = \log \sqrt{36 - x^2}$, $y = \log \left(\frac{1}{36 - x^2}\right)$, and $y = \log \left(\frac{1}{\sqrt{36 - x^2}}\right)$.

180. Differentiation of Logarithmic Functions. We now proceed to find the derivative of the logarithmic function. This is done by means of the general rule used in Chapter III.

Let $y = \log u$, where u is any function of x. We first find $\frac{dy}{du}$. Giving x an increment Δx, u takes on an increment Δu, and y takes on a corresponding increment Δy. Hence, following the four steps of the general rule, we have

I. $$y + \Delta y = \log (u + \Delta u).$$

II. $$\Delta y = \log (u + \Delta u) - \log u$$

$$= \log\left(1 + \frac{\Delta u}{u}\right). \quad \text{By Formula II, § 179}$$

III. $$\frac{\Delta y}{\Delta u} = \frac{1}{\Delta u} \log\left(1 + \frac{\Delta u}{u}\right).$$

In order to evaluate the limit when $\Delta u \rightarrow 0$, the second member of this equation is multiplied by $\frac{u}{u}$ and written in the form

$$\frac{\Delta y}{\Delta u} = \frac{1}{u}\frac{u}{\Delta u}\log\left(1+\frac{\Delta u}{u}\right)$$

$$= \frac{1}{u}\log\left(1+\frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}}. \qquad \text{By Formula III}$$

When $\Delta x \rightarrow 0$, $\Delta u \rightarrow 0$, and we have

IV. $$\frac{dy}{du} = \frac{1}{u}\log\left[\lim_{\Delta u \rightarrow 0}\left(1+\frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}}\right].$$

In order to find the value of this limit, set $\frac{u}{\Delta u} = n$; whence $\frac{\Delta u}{u} = \frac{1}{n}$, and as $\Delta u \rightarrow 0$, $n \rightarrow \infty$. Therefore

$$\lim_{\Delta u \rightarrow 0}\left(1+\frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}} = \lim_{n \rightarrow \infty}\left(1+\frac{1}{n}\right)^n = e. \qquad \text{By definition}$$

Since $\log e = 1$, we have, finally,

$$\frac{dy}{du} = \frac{1}{u}.$$

But $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx},$$

by the theorem on differentiating a function of a function. We therefore have the formula

(VI) $$\frac{d}{dx}(\log u) = \frac{\frac{du}{dx}}{u}.$$

When $u = x$, this formula reduces to

(VI*a*) $$\frac{d}{dx}(\log x) = \frac{1}{x}.$$

If some number a, different from e, is used as the base of logarithms, the derivative may be found as follows:

Suppose that $y = \log_a u$. Then, by Formula Va, § 179, $y = \log_a u = \log u \log_a e$. Since $\log_a e$ is a constant, this function may be differentiated by Formula VI, giving

$$\frac{dy}{dx} = \log_a e \frac{d}{dx}(\log u) = \frac{1}{u}\frac{du}{dx}\log_a e.$$

Hence

(VIb) $$\frac{d}{dx}(\log_a u) = \frac{1}{u}\frac{du}{dx}\log_a e.$$

If $a = 10$, $\log_{10} e = 0.434$, and Formula VIb becomes

(VIc) $$\frac{d}{dx}(\log_{10} u) = \frac{0.434}{u}\frac{du}{dx}.$$

181. Application to the Graph of $y = \log x$. We are now in a position to show that the graph of $y = \log x$ discussed in § 177 has the smooth appearance there indicated. Since $\frac{dy}{dx} = \frac{1}{x} > 0$ for all values of x for which $\log x$ is defined, the graph constantly rises. At $x = 1$, $\frac{dy}{dx} = 1$; that is, the curve crosses the x-axis at an angle of 45°. As $x \to \infty$, $\frac{dy}{dx} \to 0$; hence the curve approaches parallelism with the x-axis.

Taking the second derivative, we find that $y'' = -\frac{1}{x^2}$, which is always negative. This shows that the curve has no point of inflection and is everywhere concave downward.

EXAMPLE 1. Differentiate $y = \log\left(x^2\sqrt{2 - 3x}\right)$.

Solution. It is convenient first to simplify the expression for y as much as possible by the formulas of § 179 before differentiating. Thus

$$y = \log x^2 + \log\sqrt{2 - 3x}, \qquad \text{by Formula I}$$
$$= 2\log x + \tfrac{1}{2}\log(2 - 3x). \quad \text{By Formula III}$$

Hence, by Formula VI, $\frac{dy}{dx} = \frac{2}{x} - \frac{3}{2(2 - 3x)}$.

EXAMPLE 2. Differentiate $y = x \log x$.

Solution. To this function we first apply the product formula.

$$\frac{dy}{dx} = x\frac{d}{dx}(\log x) + \log x \frac{d}{dx}x$$
$$= x \cdot \frac{1}{x} + \log x = 1 + \log x.$$

PROBLEMS

Differentiate the following functions:

1. $y = \log(ax + b)$.
2. $y = \log\left(\frac{x^2 - 4}{x^2 + 4}\right)$.
3. $y = \log(x^2 + 2x)$.
4. $y = \log_{10}(2x - x^4)$.
5. $y = x^2 \log x$.
6. $f(x) = \log x^3$.
7. $f(x) = \log^3 x$.*
8. $f(x) = \log\left(\frac{a - x}{a + x}\right)$.
9. $f(x) = \log(x + \sqrt{1 + x^2})$.
10. $z = \log\left(\frac{1 - 2y}{1 + 2y}\right)$.
11. $s = t \log \sqrt{t}$.
12. $s = \log\left(\frac{t^2}{\sqrt{3 - 2t}}\right)$.
13. $u = \log\sqrt{\frac{1 + y}{1 - y}}$.
14. $y = \log\sqrt{(2x - 1)(2x^2 - 1)}$
15. $y = x^n \log x$.
16. $f(x) = \log\left(\frac{3x + 4}{4x + 3}\right)$.
17. $f(x) = \log\left(\frac{1 - x}{\sqrt{1 + x}}\right)$.
18. $y = \frac{\log x}{x}$.
19. $y = \log(x^4\sqrt{1 + x^2})$.
20. $y = \log_{10}(x^2 + 5x)^{\frac{1}{3}}$.

21. Find y and $\frac{dy}{dx}$ for the indicated value of x:

a. $y = \log(x\sqrt{7 - x})$, $x = 4$.
b. $y = \log\left(\frac{x^2}{6 - x^2}\right)$, $x = 2$.
c. $y = x^2 \log \sqrt{x}$, $x = 3$.
d. $y = \frac{\log x}{x^2}$, $x = 2$.
e. $y = \log_{10}(x^3 + 3)$, $x = 5$.
f. $y = \log_{10}(x\sqrt{20 - 7x})$, $x = 2$.

*Note that $\log^3 x$ means $(\log x)^3$ and is not the same as $\log x^3 = 3 \log x$.

22. Find $\frac{dy}{dx}$ for the indicated value of x:

a. $y = \log(\log x)$, $x = e$.
b. $y = \log^3 x$, $x = 3$.
c. $y = \log_{10}(2x + x^3)$, $x = 2$.
d. $y = \log_{10}(x^2 + 5x)$, $x = 1$.

23. Sketch the following curves and find the slope at each point where the curve crosses the axes of coördinates:

a. $y = \log(x + 2)$.
b. $y = \log(9 - x^2)$.
c. $y = \log(x + 3)^2$.
d. $y = \log_{10} x$.
e. $y = \log(4 - x)$.
f. $y = \log\sqrt{4 - x^2}$.
g. $y = \log\sqrt{3 - x}$.
h. $y = \log_{10}(16 - x^2)$.

24. Find the point on the curve $y = \log x^2$ where the tangent is (a) parallel to the line $x - 2y + 6 = 0$; (b) perpendicular to the line $x + y - 1 = 0$.

25. Show that all curves of the system $y = \log kx$ have the same slope; that is, the slope is independent of k.

26. Draw the graph of $y = \log x$ and its circle of curvature at the point of intersection with the x-axis (see § 140).

27. Find the maximum and minimum and inflectional points of the following curves and draw their graphs:

a. $y = \log(1 + x^2)$.
b. $y = \frac{x}{\log x}$.
c. $y = x \log x$.
d. $y = \log(8x - x^2)$.

28. If $\log 10 = 2.303$, approximate $\log 10.2$ by means of $d(\log x)$ (see § 87).

29. If $y = k \log ax$, show that the rate of increase of y with respect to x is inversely proportional to x.

182. Logarithmic Differentiation. When $y = f(x)$ is a product or quotient composed of three or more factors, it may be differentiated more easily by taking its logarithms, simplifying, and using the rule for differentiation of logarithms than by using the rules for products and quotients. This process is called differentiating logarithmically.

To illustrate, let

$$y = \frac{(x+1)^{\frac{5}{2}}(x-2)}{x^2+4}.$$

Then $\log y$ must equal the logarithm of the expression on the right-hand side of the equation. Hence

$$\log y = \tfrac{5}{2}\log(x+1) + \log(x-2) - \log(x^2+4).$$

Differentiation with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{5}{2(x+1)} + \frac{1}{x-2} - \frac{2x}{x^2+4}.$$

Multiplying both sides by y, we have

$$\frac{dy}{dx} = \left[\frac{5}{2(x+1)} + \frac{1}{x-2} - \frac{2x}{x^2+4}\right]\frac{(x+1)^{\frac{5}{2}}(x-2)}{x^2+4}$$

$$= \frac{(3x^3 - 4x^2 + 36x - 32)(x+1)^{\frac{3}{2}}}{2(x^2+4)^2}.$$

This process is convenient also when determining the maximum or minimum values for functions of x assuming a product form and involving fractional exponents. For example, to find the dimensions of the largest rectangle which can be inscribed in a circle of radius r, we have to determine the value of x making S a maximum, where $S = x\sqrt{4r^2 - x^2}$ (S is the area and x is the length of one side). Obviously S is a maximum when $\log S$ is a maximum.

$$\log S = \log x + \tfrac{1}{2}\log(4r^2 - x^2).$$

Differentiating,

$$\frac{d}{dx}(\log S) = \frac{1}{x} - \frac{x}{4r^2 - x^2} = 0.$$

Therefore

$$4r^2 - x^2 - x^2 = 0, \quad \text{or} \quad x = r\sqrt{2}.$$

PROBLEMS

Differentiate logarithmically

1. $y = (x+1)(x+2)^2(x+3)^3.$

2. $y = x\sqrt{(x+3)(2-x^2)}.$

3. $y = \dfrac{x(1+x^2)}{\sqrt{1-x^2}}.$

4. $y = \dfrac{\sqrt{x^3+2}}{x\sqrt{3-2x}}.$

Find y and $\dfrac{dy}{dx}$ for the indicated value of x:

5. $y = x\sqrt{(x+1)(x+2)},\ x = 3.$

6. $y = \sqrt{x(2x-1)(3x-2)},\ x = 2.$

7. $y = \dfrac{(x+2)(2x-1)}{x^2},\ x = 2.$

8. $y = x^2\sqrt{\dfrac{x+5}{x-5}},\ x = 7.$

183. Differentiation of the Exponential Functions. Let $y = a^u$, where u is a function of x. Taking the logarithm of both sides,

$$\log y = u \log a.$$

Solving for u,

$$u = \frac{\log y}{\log a}.$$

Differentiating with respect to y,

$$\frac{du}{dy} = \frac{1}{y \log a};$$

whence

$$\frac{dy}{du} = y \log a = a^u \log a.$$

Applying the rule for differentiating the function of a function, we now have

$$\textbf{(VII)} \qquad \frac{d}{dx}(a^u) = a^u \log a \frac{du}{dx}.$$

Special Cases. When $u = x$,

(VII a) $$\frac{d}{dx}(a^x) = a^x \log a.$$

When $a = e$, $\log a = 1$, and

(VII b) $$\frac{d}{dx}(e^u) = e^u \frac{du}{dx}.$$

When $a = e$ and $u = x$,

(VII c) $$\frac{d}{dx}(e^x) = e^x.$$

The formulas for the exponential functions must not be confused with that for the power function. In the former case the base is a constant and the exponent a variable, while in the latter the base is variable and the exponent is constant. In the unusual case where both base and exponent are variable, logarithmic differentiation should be used.

184. The Power Function. By means of the derivative of the exponential function, we are able to prove the formula $\frac{d}{dx}x^n = nx^{n-1}$, which was demonstrated in Chapter IV for the case where n is a positive integer.

Set $y = x^n$ and take the logarithms of both sides. Then

$$\log y = n \log x.$$

This gives $$y = e^{n \log x}.$$

Differentiating by Formula VII b, we have

$$\frac{dy}{dx} = e^{n \log x} \cdot \frac{d}{dx}(n \log x)$$
$$= e^{n \log x} \cdot \frac{n}{x}$$
$$= x^n \cdot \frac{n}{x}$$
$$= nx^{n-1}.$$

EXAMPLE 1. Differentiate $y = 10^{2x - x^2}$.

Solution. By Formula VII, we have

$$\begin{aligned}\frac{dy}{dx} &= 10^{2x-x^2} \cdot \log 10 \cdot \frac{d}{dx}(2x - x^2)\\ &= 10^{2x-x^2} \cdot \log 10 \cdot (2 - 2x).\end{aligned}$$

EXAMPLE 2. Differentiate $y = xe^{\frac{x}{a}}$.

Solution. By the formula for a product and by Formula VII b, we get

$$\begin{aligned}\frac{dy}{dx} &= x \cdot e^{\frac{x}{a}} \frac{d}{dx}\left(\frac{x}{a}\right) + e^{\frac{x}{a}} \frac{d}{dx}(x)\\ &= x \cdot e^{\frac{x}{a}} \frac{1}{a} + e^{\frac{x}{a}}\\ &= \left(\frac{x}{a} + 1\right) e^{\frac{x}{a}}.\end{aligned}$$

PROBLEMS

Differentiate the following functions:

1. $y = e^{ax}$.
2. $y = e^{2x-3}$.
3. $y = a^{-4x}$.
4. $f(x) = e^{a^2 - x^2}$.
5. $f(x) = a^{\log x}$.
6. $s = e^{t^2}$.
7. $y = 10^{x^2}$.
8. $y = e^x(1 - x)$.
9. $y = \dfrac{e^x - 1}{e^x + 1}$.
10. $y = \dfrac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)$.
11. $y = \log \dfrac{e^x}{1 + e^x}$.
12. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.
13. $y = x e^x$.
14. $y = e^{\frac{1}{x}}$.
15. $y = \dfrac{2}{e^{x^2}}$.
16. $s = e^t \log t$.
17. $u = \dfrac{\log w}{e^w}$.
18. $y = x^x$.

19. Show that if $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$, then $y'' = \frac{y}{a^2}$.

20. Show that the nth derivative of $y = a^x$ is $a^x \log^n a$.

21. Show that the graph of $y = e^x$ has no point of inflection.

22. What is the minimum value of $y = ae^{kx} + be^{-kx}$?

23. Show, by using differentials, that if $e^2 = 7.39$, $e^{2.1} = 8.13$, approximately.

24. Find the maximum point and the points of inflection of the graph of $y = e^{-x^2}$ and draw the curve.

25. Show that the maximum rectangle which can be inscribed under the curve in Problem 24 has two of its vertices at the points of inflection.

26. Find the minimum point of the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ and show that the curve has no points of inflection.

27. Show that the hyperbolic sine curve $y = \frac{1}{2}(e^x - e^{-x})$ has no maximum or minimum points, and find its point of inflection.

28. Find the minimum point and the point of inflection of the curve whose equation is $y = xe^x$. Draw the curve.

29. Find the radius of curvature of each of the following curves at the point of intersection with the y-axis:

a. $y = e^x$. *b.* $y = e^{-x^2}$. *c.* $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$. (See § 140)

185. Some Properties of the Exponential Function. The most striking of the formulas of § 183 is VII *c*, which shows that the operation of differentiation leaves the function e^x unchanged. This fact gives an interesting construction for the tangent to the graph of $y = e^x$ at any point. For, if α is the inclination of the tangent at any point, $\tan \alpha = \frac{dy}{dx} = e^x = y$. But, since the tangent of any acute angle in a right triangle is the quotient of the opposite

side over the adjacent, the tangent to the curve must cut the x-axis at a point one unit to the left of the foot of the ordinate, as the adjoining figure shows.

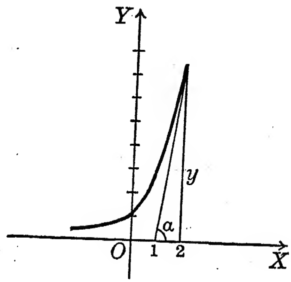


FIG. 196

Since tables of powers of e are always available, the general exponential function a^x is usually written in the form e^{kx}. This can always be done, since any constant $a = e^k$, where $k = \log a$. Hence $a^x = (e^k)^x = e^{kx}$. For example, $10^x = e^{2.30259x}$. Note that if $a < 1$, k is negative. We shall therefore discuss $y = Ce^{kx}$ as the most general exponential function, the constant C being introduced so as to permit the discussion to cover cases where y is proportional to e^{kx}. Obviously C is the value of y corresponding to $x = 0$.

Differentiating, $\dfrac{dy}{dx} = kCe^{kx}$. Hence $\dfrac{dy}{dx} = ky$, or $\dfrac{dy/dx}{y} = k$.

We thus see that the rate of change of this function with respect to x is proportional to the function itself and that the factor of proportionality is the coefficient of x in the exponent. When y decreases, k is negative. For any function the quotient $\dfrac{dy/dx}{y}$ is called the *relative rate of change*. Hence the exponential function has a constant relative rate of change.

EXAMPLE. If a heated object is cooled in moving air, and if T is the varying difference of temperature between the heated object and the air, T is given by $T = T_0 e^{-kt}$, where t is the time and T_0 and k are constants (Newton's Law of Cooling). It is found by experiment that T falls from 40° C. to 31° C. in 3 min. Find T_0 and k, show that the rate of cooling is proportional to

the temperature at any time, and find the temperature and the rate of cooling after 6 min.

Solution. Let $t = 0$ when $T = 40$; then $T = 31$ when $t = 3$. Substituting $t = 0$, we have $40 = T_0 e^0$, or $T_0 = 40$. The given formula now reads $T = 40\,e^{-kt}$.

But when $t = 3$, $T = 31$. Hence $31 = 40\,e^{-3k}$. If we take the natural logarithms of both sides of this equation, we have $\log 31 = \log 40 - 3\,k$; whence $3\,k = \log 4 - \log 3.1 = 1.386 - 1.131 = 0.255$, or $k = 0.085$.

The complete formula is, then, $T = 40\,e^{-0.085\,t}$.

Differentiating with respect to t, we find

$$\frac{dT}{dt} = 40\,e^{-0.085\,t}(-0.085) = -0.085\,T.$$

Hence the rate of cooling is proportional to the temperature.

When $t = 6$, $T = 40e^{-0.51} = 24°$, nearly. The rate of cooling is $\frac{dT}{dt} = -0.085 \times 24 = 2.04°$ per minute at this instant.

186. Relation between the Exponential Function and Compound Interest. Let us recall from arithmetic and algebra the method of calculating compound interest. Suppose that \$1000 is deposited in a savings bank paying 4 per cent compounded annually. At the end of the first year the amount due the depositor will have increased by 4 per cent of itself and will be \$1000 (1.04) = \$1040. During the second year there is a 4 per cent increase added to the \$1040, and at the close the amount due is \$1040 (1.04) = \$1081.60. In general, if A_0 is the amount in the beginning, i the rate of interest, and A the amount after t years, we have $A = A_0(1 + i)^t$.

At first sight it seems that we have here a method of increase precisely the same as that of the exponential function, since the yearly increment is a constant (i) times the amount due at the beginning of the year. There is a

distinct difference, however. In the case of the exponential function the increase is continuous, whereas in the case of the savings bank no interest is paid except at yearly intervals. Thus, in the special example considered, A remains equal to \$1000 until $t=1$; then its value jumps to \$1040 and remains constant until $t=2$, etc.

On the other hand, if interest is compounded semiannually, 2 per cent is added each half-year, and the successive amounts will be as follows:

$$\begin{aligned}
t&=0.5, & A&=1000\,(1.02) = 1020;\\
t&=1, & A&=1000\,(1.02)^2 = 1040.40;\\
t&=1.5, & A&=1000\,(1.02)^3 = 1061.21;\\
t&=2, & A&=1000\,(1.02)^4 = 1082.43.
\end{aligned}$$

In general, if the interest is compounded k times per year, the amount is increased by $\frac{i}{k}$ times itself every kth part of a year, and we have

$$A=A_0\left(1+\frac{i}{k}\right)^{kt}.$$

For the increase to be continuous at a constant relative rate, the compounding should be instantaneous, or k should be infinite. In this case we have

$$\begin{aligned}
A &= \lim_{k\to\infty} A_0\left(1+\frac{i}{k}\right)^{kt}\\
&= A_0 \lim_{k\to\infty}\left[\left(1+\frac{i}{k}\right)^{\frac{k}{i}}\right]^{it} = A_0 e^{it},
\end{aligned}$$

since the bracket is of the form $(1+n)^{\frac{1}{n}}$ used in defining e in § 175. Thus we have shown that if a quantity increases continuously at a constant relative rate with respect to some variable, it is an exponential function of that variable.

PROBLEMS

1. Find the expression for the velocity when the distance traversed by a moving point is expressed as a function of the time by the equation

$$s = (A + Bt)e^{-kt}.$$

2. The number of bacteria in a certain culture obeyed the formula $N = 1000\,e^{0.06\,t}$, where t was measured in hours. Find the number after 2 hr., and the rate at which the number was then changing.

3. It is found that barometric pressure obeys the formula $p = Ce^{-kh}$, where C and k are constants and h is the height. If $p = 30$ in. when $h = 0$, and $p = 24$ in. when $h = 5800$ ft., find C and k, and express the rate of change of pressure in terms of p. Also find the height corresponding to a pressure of 26 in.

4. The amount of light which passes through glass is found from the fact that a fixed per cent of the total is absorbed by the glass. Show that the formula $Q = Q_0 e^{-kx}$, where Q is the quantity passing through a thickness x, satisfies the law given above.

187. The Integrals of $\dfrac{du}{u}$ and $e^u du$. The formulas for differentiation derived in this chapter give us new formulas for integration.

From Formula VI*a* we obtain

$$\text{(VIII)} \qquad \int \frac{du}{u} = \log u + C.$$

This is the exceptional case, $n = -1$, of the general power formula

$$\int u^n du = \frac{u^{n+1}}{n+1} + C,$$

which is good for all values of n except -1.

Formula VII*c* gives

$$\text{(IX)} \qquad \int e^u du = e^u + C.$$

Formula VII*a* gives

$$\text{(X)} \qquad \int a^u du = \frac{a^u}{\log a} + C.$$

EXAMPLE 1. Find the value of $\int_0^2 \frac{x\,dx}{x^2+3}$.

Solution. Let $u = x^2 + 3$; then $du = 2\,x\,dx$

and $$\int \frac{x\,dx}{x^2+3} = \frac{1}{2}\int \frac{2\,x\,dx}{x^2+3} = \frac{1}{2}\log(x^2+3).$$

Substituting the limits for x, we get

$$\int_0^2 \frac{x\,dx}{x^2+3} = \frac{1}{2}\Big[\log(x^2+3)\Big]_0^2 = \frac{1}{2}(\log 7 - \log 3)$$
$$= \tfrac{1}{2}(1.946 - 1.099) = 0.424.$$

In numerical examples of this kind it must be remembered that *natural logarithms* are used.

EXAMPLE 2. Integrate $\int \frac{x^3\,dx}{x-2}$.

Solution. Since the degree of the denominator is not higher than the degree of the numerator, we must divide the numerator by the denominator until the remainder is of lower degree than the denominator. This gives

$$\frac{x^3}{x-2} = x^2 + 2\,x + 4 + \frac{8}{x-2}.$$

Hence

$$\int \frac{x^3\,dx}{x-2} = \int x^2 dx + \int 2\,x\,dx + \int 4\,dx + \int \frac{8\,dx}{x-2}$$
$$= \frac{x^3}{3} + x^2 + 4\,x + 8\log(x-2) + C.$$

EXAMPLE 3. Find the value of $\int_0^a e^{\frac{x}{a}} dx$.

Solution. Here $u = \frac{x}{a}$ and $du = \frac{dx}{a}\cdot$ Hence

$$\int_0^a e^{\frac{x}{a}} dx = a\int_0^a e^{\frac{x}{a}} \frac{dx}{a}$$

$$= ae^{\frac{x}{a}}\Big]_0^a$$

$$= ae - a$$

$$= 1.718\, a.$$

PROBLEMS

Integrate the following:

1. $\int e^{ax} dx.$
2. $\int e^{-x} dx.$
3. $\int e^{2s} ds.$
4. $\int \frac{3\, dx}{e^x}.$
5. $\int a^{2x} dx.$
6. $\int xa^{x^2} dx.$
7. $\int \frac{x\, dx}{x^2+1}.$
8. $\int \frac{x^2 dx}{x+1}.$
9. $\int \frac{(x-1)\, dx}{x^2-2x-5}.$
10. $\int \frac{5x^2 dx}{10x^3+6}.$
11. $\int \frac{(y^2-2)^3 dy}{y^5}.$
12. $\int \frac{5\, bx\, dx}{8a-6bx^2}.$
13. $\int \frac{(\log x)^3 dx}{x}.$
14. $\int \frac{2e^x dx}{e^x+1}.$
15. $\int \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) dx.$
16. $\int (e^y + e^{-y})^2 dy.$
17. $\int (e^{-2t} - 2t)\, dt.$
18. $\int \frac{(x-3)\, dx}{6x-x^2}.$

Evaluate the following definite integrals:

19. $\int_0^1 e^{2x} dx.$
20. $\int_2^5 \frac{dx}{x+2}.$
21. $\int_1^e \frac{dx}{x}.$
22. $\int_1^2 xe^{x^2} dx.$
23. $\int_0^3 \frac{x\, dx}{x^2+1}.$
24. $\int_2^3 \frac{t\, dt}{t^2+1}.$
25. $\int_0^2 \frac{x^3 dx}{x+1}.$
26. $\int_0^1 \frac{dx}{e^{3x}}.$

27. Find the area bounded by the equilateral hyperbola $xy = 12$, the x-axis, and the lines $x = 2$ and $x = 4$.

28. Find the area bounded by the equilateral hyperbola $xy = 12$, the y-axis, and the lines $y = 3$ and $y = 6$.

29. P and Q are any two points on an equilateral hyperbola $xy = k$. Show that the area bounded by the arc PQ, the ordinates of P and Q, and the x-axis is equal to the area bounded by PQ, the abscissas of P and Q, and the y-axis.

30. Find the area in the first quadrant bounded by the line $y = \frac{x}{4}$ and the curve $y(1 + x^2) = x$.

31. Find the area bounded by the curve $y = e^x$, the y-axis, the x-axis, and any ordinate.

32. Find the area bounded by the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$, the x-axis, and the lines $x = a$ and $x = -a$.

33. Find the area bounded by the curve $(x^2 + 1)y = x$, the x-axis, and the lines $x = 1$ and $x = 4$.

34. Find the area bounded by the curve $y = e^x$, the y-axis, and the line $y = 6$.

35. Find the length of an arc of the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ extending from $x = 0$ to $x = a$.

36. Find the length of the arc of the curve whose equation is $y = \frac{x^2}{4} - \frac{1}{2}\log x$ extending from $x = 1$ to $x = 4$.

37. Find the volume generated by revolving about the x-axis the area bounded by $y^2(6 - x) = x$, $y = 0$, and $x = 4$.

38. Find the volume generated by revolving about the x-axis the area bounded by $y^2(2a - x) = x^3$, $x = a$, and $y = 0$.

39. Find the volume generated by revolving about the x-axis the area bounded by $y = e^x$, $x = 0$, $x = 1$, and $y = 0$.

40. Find the volume generated by revolving about the x-axis the loop of the curve $(x - 4)y^2 = x(x - 3)$.

41. Find the volume generated by revolving about the x-axis the area found in Problem 32.

42. Find the volume generated by revolving about the x-axis the area bounded by the curve $y = e^{-x}$, the lines $x = 0$ and $x = 10$, and the x-axis.

188. The Differential Equation $\frac{dy}{dx} = ky$. In § 185 it was shown that the relative rate of increase of the exponential function is constant; and in § 186 it was shown in an indirect manner that, conversely, any function whose relative rate of increase is constant must be an exponential function. By means of integration this latter fact may be demonstrated much more simply.

For, if $y = f(x)$ changes at a constant relative rate, $\frac{dy/dx}{y} = k$, or $\frac{dy}{dx} = ky$, where k is a positive constant if y is increasing and a negative constant if it is decreasing. Such an equation, containing derivatives or differentials, is known as a *differential equation*, and the process of expressing y as a function of x is known as *solving* it.*

Clearing of fractions, $dy = ky\,dx$. Dividing by y, we have

$$\frac{dy}{y} = k\,dx.$$

Since but one variable appears on each side, we can integrate both sides, and this gives

$$\log y = kx + C.$$

As C represents any constant, we may replace it by $\log c$, giving

$$\log y = kx + \log c$$

or

$$\log \frac{y}{c} = kx.$$

* Compare §§ 146–148.

The definition of a logarithm transforms this into

$$\frac{y}{c} = e^{kx},$$

or

$$y = ce^{kx},$$

which is the usual form.

The constant of integration, c, is determined by a pair of initial values of x and y. In solving problems of this kind the above result should not be used as a formula, but the integration should always be performed.

EXAMPLE. The number of bacteria in a culture increased at a constant percentage rate of 40 per cent, the unit of time being one hour. If there were 1000 at the start, find the formula for the number at any time.

Solution. Let t be the time, and N be the number. By the conditions of the problem, $\frac{dN}{dt} = kN$, where $k = 0.40$. Hence

$$dN = 0.4\,N\,dt,$$

or

$$\frac{dN}{N} = 0.4\,dt.$$

Integrating, $$\log N = 0.4\,t + \log c.$$

Since $$N = 1000$$

when $$t = 0,$$

$$\log 1000 = \log c,$$

or $$c = 1000.$$

Therefore $$\log N = 0.4\,t + \log 1000,$$

$$\log \frac{N}{1000} = 0.4\,t,$$

or $$N = 1000\,e^{0.4\,t}.$$

When $t = 5$, $N = 7389$. If the number were calculated according to compound interest, we should have $N = 1000\,(1.4)^5 = 5378$.

PROBLEMS

1. The slope of a curve at every point is $\frac{2x}{1+x^2}$, and it passes through the point (0, 0). Find its equation.

2. Find the equation of the system of curves whose slope at any point is $\frac{-y}{x}$.

3. Find the equation of the system of curves whose slope at any point is xy.

4. Which curve in Problem 3 passes through (2, 1)?

5. Find the most general function which is unchanged by differentiation.

6. Given that the intensity of light decreases 2 per cent by passing through 1 mm. of glass, find the intensity I as a function of x, the thickness of glass through which the light passes.

7. In Problem 6 find the thickness of glass which would absorb one half the light.

8. The speed (v) of a rotating wheel after the power was shut off was decreased at a rate (per second) which was at every instant 25 per cent of v itself. Find a formula for v in terms of t if v was originally 1000 revolutions per second. Find the value of v after 10 sec.

9. Atmospheric pressure varies thus with the height above sea level: $\frac{dp}{dh} = -0.00004p$, where p is measured in pounds per square inch and h is measured in feet. Find a formula for p at any height, knowing $p = 15$ lb. when $h = 0$.

10. When an iron rod is heated, its length increases thus: $\frac{dL}{dT} = 0.00001\,L$. Express L as a function of T, if $L = 60$ in. when $T = 0°$ C.

11. A boat moving in still water is subject to a retardation proportional to its velocity at any instant. Show that the velocity t seconds after the power is shut off is given by the formula $v = ce^{-kt}$, where c is the velocity at the instant the power is shut off.

CHAPTER XV

THE TRIGONOMETRIC FUNCTIONS

189. Preliminary Remarks. The solution of many problems, such as some of those involving curvilinear motion, requires the analytic use of trigonometric functions. Furthermore, many algebraic functions can be integrated only by means of trigonometric functions. We therefore turn to a discussion of them.

In the analytic use of the trigonometric functions a thorough familiarity with the usual formulas connecting these functions is necessary, and this knowledge is assumed in what follows.

Of the many different units of angular measurement the most familiar is that in which a right angle is divided into ninety degrees, each degree into sixty minutes, and each minute into sixty seconds. For purposes of plotting the graphs of trigonometric functions, measuring slopes, rates, areas, volumes, etc., it is necessary to use radian measure. It must be clearly understood that a radian and a degree are merely two different units for measuring the same quantity. A radian is a central angle whose intercepted arc equals the radius of the circle. From this definition it follows that $\pi (= 3.1416)$ radians $= 180°$, 1 radian $= 57°.30$, $1° = 0.0175$ radians, etc. A table of equivalents for radians and degrees is contained in any good set of tables. These equivalents are all approximate, owing to the fact that the circumference and the radius of a circle are incommensurable.

Since by definition a central angle of one radian intercepts an arc of length one radius, it follows at once that a central angle of θ radians intercepts an arc of θ radii. That is,

$$s = \theta r,$$

where θ is the value of the central angle in radians, and r and s are respectively the lengths of the radius and the intercepted arc measured in the same linear units.

x	x	$y =$	$y =$
Degrees	*Radians*	$\sin x$	$\cos x$
0	0.00	0.00	1.00
15	0.26	0.26	0.97
30	0.52	0.50	0.87
45	0.79	0.71	0.71
60	1.05	0.87	0.50
75	1.31	0.97	0.26
90	$\frac{\pi}{2} = 1.57$	1.00	0.00
180	$\pi = 3.14$	0.00	-1.00
270	$\frac{3\pi}{2} = 4.71$	-1.00	0.00
360	$2\pi = 6.28$	0.00	1.00
Etc.	Etc.	Etc.	Etc.

190. Graphs of $y = \sin x$ and $y = \cos x$. In plotting x is always measured in radians. If tables are available which give the values of the sine and cosine for angles measured in radians, the corresponding values of x and y may be read off directly from the tables. If it is necessary to use the ordinary tables for angles measured in degrees, it is best to make three columns, — the first giving the values of x in degrees, the second the corresponding values in radians taken from the conversion table, and the third the values of $\sin x$ or $\cos x$. This is done in the table given above.

Since $\sin(-x) = -\sin x$ and $\sin(x \pm 2\pi) = \sin x$, it is unnecessary to continue the table for the sine curve beyond the range 0–360°, or 0–2π radians. The complete curve consists simply of an indefinite repetition of this portion along both sides of the x-axis. The same scale is always used for x and y.

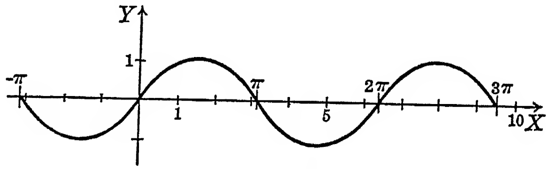


FIG. 197

The graph of $y = \cos x$ may be plotted from the table of values, but this is not necessary. For the formula $\sin\left(x + \frac{\pi}{2}\right) = \cos x$ shows that the graph of $y = \sin x$ becomes that of $y = \cos x$ if the origin is moved to the point $\left(\frac{\pi}{2}, 0\right)$.

These graphs should be known well enough so that they can be reproduced from memory. They are of considerable

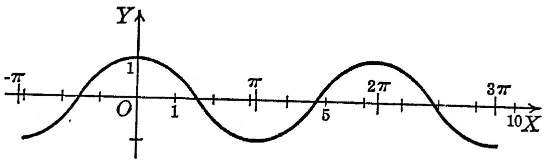


FIG. 198

aid in recalling many details regarding the functions $\sin x$ and $\cos x$. It should be noted that $\sin x$ equals 0 when $x = \pm n\pi$; that it has its maximum value $(+1)$ when $x = \frac{\pi}{2} \pm 2n\pi$; that it has its minimum value (-1) when $x = \frac{3\pi}{2} \pm 2n\pi$. It should be noted that $\cos x$ equals 0 when

$x = \frac{\pi}{2} \pm n\pi$; that it has its maximum value $(+1)$ when $x = 2n\pi$; and that it has its minimum value (-1) when $x = \pi \pm 2n\pi$.

191. Periodicity, Range of Values, and Multiple Angles. Functions whose values are repeated after the independent variable has passed through a certain range of values are called *periodic*, and may be formally defined as follows: *The function $f(x)$ is periodic if there is a constant k such that $f(x+k) = f(x)$ for all values of x.* The constant k is called the *period*. As mentioned in the previous section, $\sin x$ and $\cos x$ have the period 2π.

NOTE. Obviously, if k is a period of $f(x)$, then $2k$, $3k$, etc. are periods. In this and all that follows, it is assumed that k is the smallest period. It can be shown that no other constant not a multiple of k can be a period of $f(x)$.

The range of values of the sine and cosine is from -1 to $+1$. If we multiply a function by a constant, the range of values of the function is multiplied by the same constant. Thus $y = 3\sin x$ varies from -3 to $+3$.

Let us now investigate the effect upon the period of multiplying the independent variable by a constant. We have the following theorem:

Theorem. *If $f(x)$ has the period k, the period of $f(nx)$ is $\frac{k}{n}$.*

Proof. We must show that $f(nx)$ is unchanged if x is replaced by $x + \frac{k}{n}$. Substituting, we have

$$f\left(n\left[x + \frac{k}{n}\right]\right) = f(nx + k) = f(nx),$$

since k is a period of f.

Thus, if the independent variable is multiplied by a constant n, the period is divided by the same constant. For example, the period of $y = \sin 3x$ is $\frac{2\pi}{3} = 120°$.

Any curve whose equation is of the type $y = a \sin nx$ or $y = a \cos nx$ can be sketched by the aid of these properties and the knowledge of the graphs of $y = \sin x$ and $y = \cos x$. The method of sketching such a graph is as follows:

1. Determine the period.
2. Plot the points on the curve at each quarter period.
3. These points are the intercepts and the maximum or minimum points, by means of which the curve may be drawn in.

EXAMPLE 1. Sketch the graph of $y = \frac{3}{2} \sin \frac{x}{2}$.

Solution. The period is $\frac{2\pi}{\frac{1}{2}} = 4\pi$.

The values of y at each quarter period are given in the table.

x	y
0	0
π	$\frac{3}{2}$
2π	0
3π	$-\frac{3}{2}$
4π	0

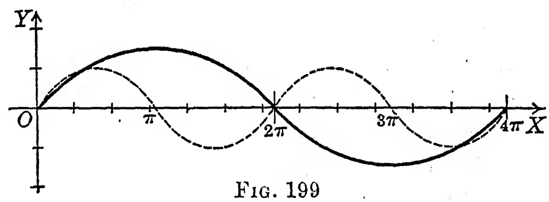


FIG. 199

The graph is drawn in the figure, where the graph of $y = \sin x$ is also shown by a dotted curve for the purpose of comparison.

EXAMPLE 2. Sketch the graph of $y + 2 \cos \frac{\pi x}{2} = 0$.

Solution. The period is $\frac{2\pi}{\frac{\pi}{2}} = 4$.

The values of y at each quarter period are given in the table.

x	y
0	-2
1	0
2	2
3	0
4	-2

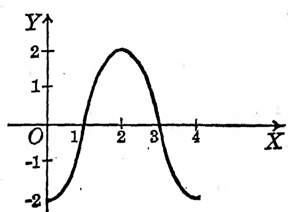


FIG. 200

PROBLEMS

1. Make out a table of values for $y = \cos x$, with x ranging through a complete period, and make a careful drawing of the graph.

2. What symmetry have the graphs of $y = \sin x$ and $y = \cos x$? Prove your statements.

3. In each of the following equations find the period and the range of values, and sketch the graph for a complete period:

a. $y = \sin \dfrac{x}{3}$.

b. $y = \cos \dfrac{x}{2}$.

c. $y = \sin 3x$.

d. $y = 2 \cos x$.

e. $y = 3 \sin 2x$.

f. $y = \cos \pi x$.

g. $y = 4 \sin \dfrac{\pi x}{4}$.

h. $y = 3 \cos \dfrac{\pi x}{2}$.

i. $y + \sin \pi x = 0$.

j. $y + \cos 2\pi x = 0$.

k. $y + 2 \sin \dfrac{x}{2} = 0$.

l. $y + \cos 2x = 0$.

4. Make a table of values for the range indicated and plot the graph of each of the following equations:

a. $y = x \sin x$. $(0 \text{ to } 2\pi)$

b. $10y = x^2 \sin x$. $(0 \text{ to } 2\pi)$

5. Make a table of values for the range indicated and plot the graph of each of the following equations:

a. $y = \sin\left(x + \dfrac{\pi}{4}\right)$. $(0 \text{ to } 2\pi)$

b. $y = \cos\left(x - \dfrac{\pi}{4}\right)$. $(0 \text{ to } 2\pi)$

c. $y = \sin \pi (x + \frac{1}{4})$. $(0 \text{ to } 2)$

d. $y = \cos\left(\dfrac{\pi x}{2} + \dfrac{\pi}{4}\right)$. $(0 \text{ to } 4)$

6. Determine which of the following functions are periodic, and find their periods:

a. $y = x + 2 \sin x$.

b. $y = \cos \dfrac{x}{2} + \sin 2x$.

c. $y = x - \cos \dfrac{\pi x}{2}$.

d. $y = \sin \pi x + \cos \pi x$.

192. The Limit of $\frac{\sin\theta}{\theta}$ as θ approaches 0. In order to differentiate the sine function, we must first evaluate this limit. In the figure the angle 2θ is drawn with its vertex at the center of a circle of radius 1. Let OC bisect the angle and let AC and BC be tangent to the circle at A and B respectively.

By plane geometry,

$$\text{chord } AB < \text{arc } ATB < AC + BC.$$

It is easy to show that since the radius of the circle is 1,

$$BS = AS = \sin\theta$$

and $BC = AC = \tan\theta$.

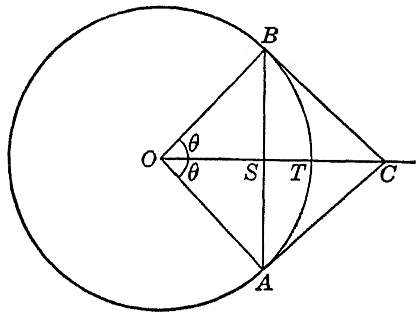


FIG. 201

But if the angle 2θ is measured in radians, arc $ATB = 2\theta$, by § 189.

Substituting these values above, we have

$$2\sin\theta < 2\theta < 2\tan\theta.$$

Dividing by $2\sin\theta$, we get

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Now as $\theta \rightarrow 0$, $\cos\theta \rightarrow 1$; hence $\frac{1}{\cos\theta} \rightarrow 1$. Since $\frac{\theta}{\sin\theta}$ is in value always between 1 and $\frac{1}{\cos\theta}$, it must also approach 1 as a limit. Hence

$$\lim_{\theta\rightarrow 0}\frac{\theta}{\sin\theta} = 1$$

and therefore

$$\lim_{\theta\rightarrow 0}\frac{\sin\theta}{\theta} = 1.$$

Note that this relation holds only if θ is measured in radians. As stated before, it is essential for the next section; but it has also an importance of its own, since it shows that, for small values of θ, $\sin\theta$ is nearly equal to θ itself.

193. The Derivative of $\sin u$. We first differentiate with respect to u. The general rule gives at once the following equations:

$$\begin{aligned} y &= \sin u, \\ y + \Delta y &= \sin(u + \Delta u), \\ \Delta y &= \sin(u + \Delta u) - \sin u \\ &= 2\cos\left(u + \frac{\Delta u}{2}\right)\sin\frac{\Delta u}{2}. \end{aligned}$$

The last equality is given by the trigonometric formula

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}.$$

Dividing by Δu, we obtain

$$\begin{aligned} \frac{\Delta y}{\Delta u} &= \frac{2}{\Delta u}\cos\left(u + \frac{\Delta u}{2}\right)\sin\frac{\Delta u}{2} \\ &= \cos\left(u + \frac{\Delta u}{2}\right)\cdot\left(\frac{\sin\frac{\Delta u}{2}}{\frac{\Delta u}{2}}\right). \end{aligned}$$

As $\Delta u \to 0$, $\lim \cos\left(u + \frac{\Delta u}{2}\right) = \cos u$. The fraction in the parentheses is in the form $\frac{\sin\theta}{\theta}$ considered in the previous section; hence its limit is 1. We therefore have

$$\frac{dy}{du} = \cos u.$$

But
$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}.$$

Hence

(I) $$\frac{d}{dx}\sin u = \cos u\frac{du}{dx}.$$

194. The Derivative of $\cos u$. Since $\sin\left(u+\frac{\pi}{2}\right)=\cos u$, we can use the formula of the previous section.

$$\frac{d}{dx}(\cos u)=\frac{d}{dx}\sin\left(u+\frac{\pi}{2}\right)$$

$$=\cos\left(u+\frac{\pi}{2}\right)\frac{d}{dx}\left(u+\frac{\pi}{2}\right)$$

$$=-\sin u\frac{du}{dx}.$$

The formula is then

$$\text{(II)}\qquad \boldsymbol{\frac{d}{dx}(\cos u)=-\sin u\frac{du}{dx}}.$$

Example 1. Differentiate $y=\cos^3\left(\frac{x}{2}\right)$.

Solution. Applying the formula for differentiating u^n, we have

$$\frac{dy}{dx}=3\cos^2\left(\frac{x}{2}\right)\frac{d}{dx}\cos\left(\frac{x}{2}\right)$$

$$=3\cos^2\left(\frac{x}{2}\right)\cdot\left(-\sin\frac{x}{2}\right)\cdot\frac{d}{dx}\left(\frac{x}{2}\right)$$

$$=-\frac{3}{2}\cos^2\left(\frac{x}{2}\right)\cdot\sin\frac{x}{2}.$$

Example 2. Find the maximum and minimum points and the points of inflection of $y=x-\sin 2x$ between 0 and π, and plot the graph.

Solution. Differentiating, we have

$$\frac{dy}{dx}=1-2\cos 2x,$$

$$\frac{d^2y}{dx^2}=4\sin 2x.$$

Setting the first derivative equal to zero gives $\cos 2x = \frac{1}{2}$, or $2x = 60°$ or $300°$. Hence the critical values are $x = 30° = \frac{\pi}{6}$ and $x = 150° = \frac{5\pi}{6}$. Testing these by substitution in the second derivative, we find that $x = \frac{\pi}{6}$ gives a minimum value

$$y = \frac{\pi}{6} - \sin\frac{\pi}{3}$$
$$= 0.5236 - 0.8660$$
$$= -0.3424;$$

and $x = \frac{5\pi}{6}$ gives a maximum value

$$y = \frac{5\pi}{6} - \sin\frac{5\pi}{3}$$
$$= 2.6180 + 0.8660$$
$$= 3.4840.$$

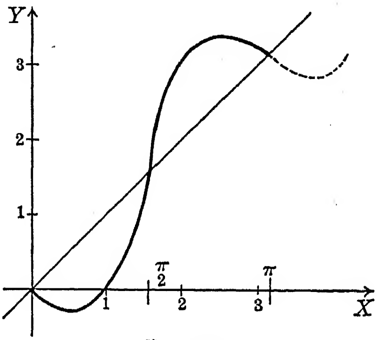


FIG. 202

Hence the minimum point is $\left(\frac{\pi}{6}, -0.3424\right)$, and the maximum point is $\left(\frac{5\pi}{6}, 3.4840\right)$.

Setting the second derivative equal to zero gives $\sin 2x = 0$; whence $x = 0°$, $90°$, or $180°$. Calculating the corresponding values of y, we find that the points of inflection are $(0, 0)$, $\left(\frac{\pi}{2}, 1.5708\right)$, $(\pi, 3.1416)$.

PROBLEMS

1. Differentiate the following functions:

a. $y = \sin 2x$.

b. $s = \cos ax$.

c. $r = a\cos 2t$.

d. $f(\theta) = a(1 - \cos\theta)$.

e. $f(y) = \sin 2y\cos 3y$.

f. $f(\theta) = \theta\sin\theta + \cos\theta$.

g. $f(x) = \sin^2 x$.

h. $f(x) = \cos^2 x^2$.

i. $f(x) = \log\cos x$.

j. $y = \log\sqrt{\sin x}$.

k. $y = e^{\sin x}$.

l. $y = \sin\sqrt{x}$.

m. $r = \sqrt{\cos\theta}$.

n. $y = x^2\sin 2x$.

o. $s = e^{-t}\cos t$.

p. $y = \frac{\sin x}{x}$.

q. $y = \frac{x}{\cos x}$.

2. In the following equations find the value of $\frac{dy}{dx}$ for the given value of x:

a. $y = 2 \sin x, \quad x = 1.$

b. $y = \cos 4x, \quad x = 0.5.$

c. $y = \log \sin x, \quad x = \frac{\pi}{4}.$

d. $y = e^{-\frac{x}{2}} \cos x, \quad x = 2.$

e. $y = x \sin \frac{x}{2}, \quad x = 2.$

f. $y = e^{\cos x}, \quad x = \frac{\pi}{2}.$

3. Find the first and second derivatives of the following functions:

a. $y = \sin \frac{x}{3}.$

b. $r = 2 \cos 3\theta.$

c. $y = x \cos 2x.$

d. $s = e^{-t} \sin 2t.$

4. Find, by differentiating, the maximum and minimum points of the following curves:

a. $y = \sin x.$

b. $y = \cos x.$

5. Find the points of inflection of

a. $y = \sin x.$

b. $y = \cos x.$

6. Find the equation of the tangent to each of the following curves at the given point:

a. $y = \sin x, \quad x = 0.$

b. $y = \cos x, \quad x = \frac{\pi}{2}.$

7. The equation of motion of a point that moves along a straight line is $s = 2 \cos 2t$. Find the position, velocity, and acceleration when $t = 0, \frac{\pi}{4}, \frac{\pi}{2}.$

8. The equation of motion of a point that moves along a straight line is $s = 3 \sin \frac{\pi t}{2}$. Find the position, velocity, and acceleration when $t = 0, 1, 2, 3, 4.$

9. Find the speed and the acceleration of a particle whose displacement x has one of the following values. In each case show that the acceleration is proportional to x.

a. $x = \sin 2t.$

b. $x = A \cos kt + B \sin kt.$

c. $x = \sin\left(\frac{t}{2} - \frac{\pi}{4}\right).$

d. $x = a \sin(kt + e).$

10. Find the angles which the curves $y = \sin x$ and $y = \cos x$ make with each other at their points of intersection.

11. Find the angles of intersection of the graphs of $y = x$ and $y = x - \sin 2x$ at the points where $x = 0$ and $x = \frac{\pi}{2}$.

12. Find the maximum, minimum, and inflectional points for the range indicated and plot the curves:

a. $y = x + 2\sin x$. $(0 \text{ to } 2\pi)$
b. $y = x - 2\sin x$. $(0 \text{ to } 2\pi)$
c. $y = x + \sin 2x$. $(0 \text{ to } \pi)$
d. $y = x + \cos 2x$. $(0 \text{ to } \pi)$
e. $y = \frac{x}{2} - \sin x$. $(0 \text{ to } 2\pi)$
f. $y = \sin x + \cos x$. $(0 \text{ to } 2\pi)$
g. $y = \frac{x}{2} + \cos x$. $(0 \text{ to } 2\pi)$
h. $y = \sin \pi x - \cos \pi x$. $(0 \text{ to } 2)$

13. Find the value of the differential of $\sin x$ when $x = 60°$ and $dx = 2'$.

14. Given $\sin 60° = 0.86603$ and $\cos 60° = 0.5$, use differentials to compute the values of the following functions to four decimal places:

a. $\sin 62°$. *b.* $\cos 61°$. *c.* $\sin 59°$. *d.* $\cos 58°$.

15. Two sides of a triangle are 3 ft. and 4 ft. respectively, and the included θ is 60°. Express the third side y as a function of θ, find its value for $\theta = 60°$, and by means of differentials find the error in y caused by an error of $3'$ in measuring θ.

195. Integrals of Sine and Cosine. The formulas for integrating $\sin u$ and $\cos u$ are obtained immediately by inverting those for differentiating. They are

$$\int \sin u\,du = -\cos u + C,$$

$$\int \cos u\,du = \sin u + C.$$

PROBLEMS

1. Integrate:

a. $\int \sin 2x\,dx.$

b. $\int \cos \frac{x}{2}\,dx.$

c. $\int (\sin x + \cos x)\,dx.$

d. $\int \sin^3 x \cos x\,dx.$

e. $\int \frac{\cos x\,dx}{\sin x}.$

f. $\int x \sin x^2\,dx.$

g. $\int e^{\cos x} \sin x\,dx.$

h. $\int \cos \log x \frac{dx}{x}.$

i. $\int (x + \sin 2x)\,dx.$

j. $\int \cos(\pi - x)\,dx.$

2. Evaluate

a. $\int_0^{\pi} \sin x\,dx.$

b. $\int_0^1 \cos \frac{\pi x}{2}\,dx.$

c. $\int_0^1 (x + \sin 2x)\,dx.$

d. $\int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x}\,dx.$

3. Sketch each of the following curves and find the area of one arch:

a. $y = 2\cos x.$

b. $y = 2\sin \frac{\pi x}{2}.$

c. $y = \cos 2x.$

d. $y = \sin \frac{x}{2}.$

4. Find the volume generated when the area bounded by the x-axis and one arch of the curve $y = \sin x$ is revolved about the x-axis.

HINT. $2\sin^2 x = 1 - \cos 2x.$

5. Find the length of one arch of the curve $y = \sin x$. Use Simpson's rule, taking $n = 4$.

6. Find the area of the surface of the solid described in Problem 4. Use Simpson's rule, taking $n = 4$.

7. Find the area in the first quadrant bounded by the y-axis, the curve $y = \sin x$, and the curve $y = \cos x$.

8. Find the area between $x = 0$ and $x = 2\pi$ which is inclosed by the curves $y = \sin x$ and $y = \cos x$.

9. Find the area bounded by the curve $y = x + \cos\frac{\pi x}{4}$, the coördinate axes, and the line $x = 2$.

10. The velocity of a point moving in a straight line is $v = 4\cos\frac{t}{2}\cdot$ Find the distance from the starting-point at the end of $\frac{\pi}{2}$ sec.

11. The acceleration of a point moving in a straight line is $a = -16\cos 2t$. The point starts from rest. Find its distance from the starting-point at the end of $\frac{\pi}{2}$ sec.

196. Graphs of the Tangent, Cotangent, Secant, and Cosecant Functions. The graphs of these functions are plotted

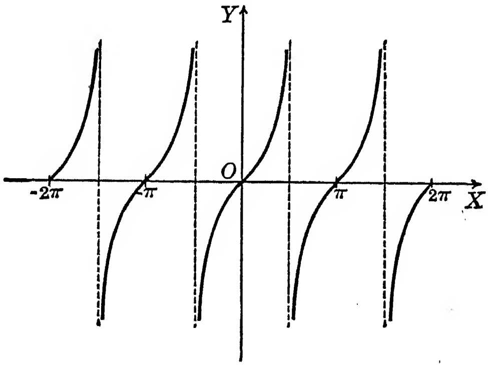


FIG. 203

in much the same way as the sine and cosine curves. Since they are all infinite for certain values of x and are periodic, they will have an infinite set of vertical asymptotes and will be made up of an infinite number of branches.

The sketching of the graphs is much simplified by recalling the relations between these functions and the sine and cosine. Thus $y = \tan x = \dfrac{\sin x}{\cos x}$ shows that $\tan x = 0$ when $\sin x = 0$, and $\tan x = \infty$ when $\cos x = 0$. Also $y = \sec x = \dfrac{1}{\cos x}$

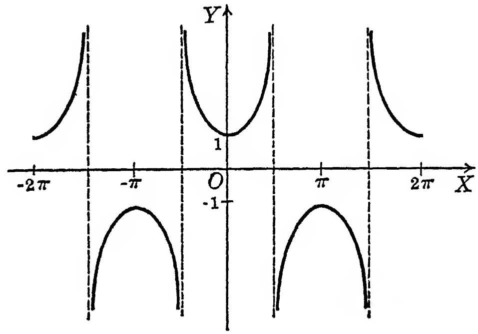


FIG. 204

shows that $\sec x = \infty$ when $\cos x = 0$; and since $\cos x$ is never greater than unity, $\sec x$ is never less than unity. The lines $x = \dfrac{\pi}{2} \pm n\pi$ are asymptotes for $y = \tan x$ and $y = \sec x$.

The facts in the table below should be verified by the student. The graphs of $y = \tan x$ and $y = \sec x$ are given; those of $y = \cot x$ and $y = \csc x$ should be plotted from a table of values and kept for reference.

FUNCTION	PERIOD	x-INTERCEPTS	ASYMPTOTES	RANGE OF VALUES
$\tan x$	π	$x = \pm n\pi$	$x = \dfrac{\pi}{2} \pm n\pi$	$-\infty$ to $+\infty$
$\cot x$	π	$x = \dfrac{\pi}{2} \pm n\pi$	$x = \pm n\pi$	$-\infty$ to $+\infty$
$\sec x$	2π	None	$x = \dfrac{\pi}{2} \pm n\pi$	$-\infty$ to -1 and $+1$ to $+\infty$
$\csc x$	2π	None	$x = \pm n\pi$	$-\infty$ to -1 and $+1$ to $+\infty$

197. Derivatives of the Tangent, Cotangent, Secant, and Cosecant. Since

$$\tan u = \frac{\sin u}{\cos u},$$

we have

$$\frac{d}{dx}(\tan u) = \frac{\cos u \dfrac{d}{dx}(\sin u) - \sin u \dfrac{d}{dx}(\cos u)}{\cos^2 u}$$

$$= \frac{\cos^2 u + \sin^2 u}{\cos^2 u}\frac{du}{dx}$$

$$= \frac{1}{\cos^2 u}\frac{du}{dx} = \sec^2 u \frac{du}{dx}.$$

Hence

(III) $$\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}.$$

Similarly,

(IV) $$\frac{d}{dx}(\cot u) = -\csc^2 u \frac{du}{dx}.$$

Setting $\sec u = \dfrac{1}{\cos u}$, we find

(V) $$\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx}.$$

Similarly,

(VI) $$\frac{d}{dx}(\csc u) = -\csc u \cot u \frac{du}{dx}.$$

The following example illustrates the use of trigonometric functions in applications.

A number of iron brackets in the form of a capital Y are to be constructed. The height of the Y is to be 12 in., and the width across the top is to be 10 in. What shape requires the least material?

The total length of the stem and the two branches must evidently be a minimum. Let $2x$ be the angle

between the branches. Then $BC = 5 \csc x$, $BD = 5 \cot x$, and $BA = 12 - 5 \cot x$. Hence the length required for any shape is
$$L = 12 - 5 \cot x + 10 \csc x.$$

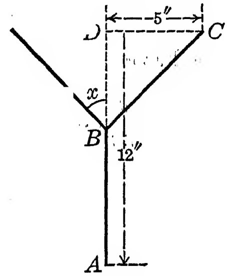


Fig. 205

Differentiating, we have
$$\frac{dL}{dx} = 5 \csc^2 x - 10 \csc x \cot x.$$

Setting this equal to 0 and dividing out $5 \csc x$ (which cannot equal 0),
$$\csc x - 2 \cot x = 0.$$
$$\csc x = 2 \cot x,$$
or
$$\frac{1}{\sin x} = \frac{2 \cos x}{\sin x};$$
whence
$$\cos x = \tfrac{1}{2}.$$

Therefore $x = 60°$ gives either a maximum or a minimum length. That the length is a minimum can be proved by getting $\frac{d^2L}{dx^2}$. This problem can be solved without recourse to trigonometric functions, but their use avoids the necessity of troublesome radicals.

PROBLEMS

1. Make a table of values, for x ranging from $-\pi$ to 2π, and plot the graphs of the following equations:

a. $y = \tan x$.
b. $y = \cot x$.
c. $y = \sec x$.
d. $y = \csc x$.

2. Indicate the relation between the corresponding ordinates of

a. $y = \sin x$ and $y = \csc x$.
b. $y = \cos x$ and $y = \sec x$.
c. $y = \tan x$ and $y = \cot x$.

3. Differentiate the following functions:

a. $f(x) = \tan ax$. *d.* $y = 2\sec\frac{x}{2}$. *g.* $F(\theta) = \tan\theta - \theta$.

b. $y = \tan(ax + b)$. *e.* $y = \csc\frac{1}{x}$. *h.* $f(x) = \log\tan x$.

c. $s = \cot t^2$. *f.* $r = \tan^2\theta$. *i.* $f(x) = \tan\log x$.

4. In the following equations find the value of the derivative for the given value of x:

a. $y = \tan\frac{\pi x}{4}$, $x = 1$. *c.* $y = x\tan x$, $x = 1$.

b. $y = x\cot x$, $x = \frac{\pi}{4}$. *d.* $y = \sec\frac{x}{3}$, $x = \pi$.

5. Write out in full the derivation of the formulas for differentiating (*a*) $\cot u$; (*b*) $\csc u$.

6. Find the slopes of $y = \tan x$ and $y = x^3$ at the origin. Sketch both curves on the same axes.

7. By differentiating the formula $\sin 2x = 2\sin x\cos x$, obtain the formula for $\cos 2x$.

8. Sketch the graph of $y = 1 + x^2$ and the branch of $y = \sec x$ cutting the y-axis.

9. Find the angles between the curves $y = \tan x$ and $y = \cot x$ at their points of intersection.

10. Find the maximum and minimum values of $2x - \tan x$ for x between 0 and π.

11. The distance of the head of a piston from the center of the driving-shaft is given by the formula

$$y = k + r\cos\theta + \sqrt{a^2 - r^2\sin^2\theta},$$

where a is the length of the connecting rod, k is another constant, r is the length of the crank, and θ is the angle the crank makes with the horizontal. If the crank rotates at a constant angular velocity of 600 revolutions per minute, find the rate at which the piston is moving when $\theta = 90°$; when $\theta = 0°$.

12. Find the maximum rectangle inscribed in a circle, using trigonometric functions.

13. Find the rectangle of maximum perimeter inscribed in a circle, using trigonometric functions.

14. A steel girder 30 ft. long is carried along a passage 10 ft. wide and into a corridor at right angles to the passage. The thickness of the girder being neglected, how wide must the corridor be in order that the girder may go round the corner?

15. A tapestry 7 ft. in height is hung on a wall so that its lower edge is 9 ft. above an observer's eye. At what distance from the wall should he stand in order to obtain the most favorable view; that is, so that the vertical angle subtended by the tapestry at his eye is a maximum?

16. A body of weight W is dragged along a horizontal plane by means of a force P whose line of action makes an angle x with the plane. The magnitude of the force is given by the equation

$$P = \frac{mW}{m \sin x + \cos x},$$

in which m denotes the coefficient of friction. Show that the pull is least when $\tan x = m$.

17. A revolving light sending out a bundle of parallel rays is at a distance of $\frac{1}{2}$ mi. from the shore and makes 1 revolution a minute. Find how fast the light is traveling along the straight beach when at a distance of 1 mi. from the nearest point of the shore.

18. An angle is increasing at a constant rate. Show that the tangent and the sine are increasing at the same rate when the angle is zero, and that the tangent increases eight times as fast as the sine when the angle is 60°.

198. Formulas of Integration of Trigonometric Functions. The six formulas of differentiation may be at once inverted into the following:

(VII) $$\int \sin u\,du = -\cos u + C.$$

(VIII) $$\int \cos u\,du = \sin u + C.$$

(IX) $$\int \sec^2 u\,du = \tan u + C.$$

(X) $$\int \csc^2 u\,du = -\cot u + C.$$

(XI) $$\int \sec u \tan u\,du = \sec u + C.$$

(XII) $$\int \csc u \cot u\,du = -\csc u + C.$$

To integrate $\tan u\,du$, we proceed as follows:

$$\int \tan u\,du = \int \frac{\sin u\,du}{\cos u} = -\int \frac{d(\cos u)}{\cos u}$$
$$= -\log \cos u + C = +\log \frac{1}{\cos u} + C$$
$$= \log \sec u + C.$$

Hence

(XIII) $$\int \tan u\,du = \log \sec u + C.$$

Likewise,

(XIV) $$\int \cot u\,du = \log \sin u + C.$$

To integrate $\sec u\,du$, we multiply numerator and denominator by $\sec u + \tan u$. Then

$$\int \sec u\,du = \int \frac{(\sec^2 u + \sec u \tan u)\,du}{\sec u + \tan u}$$
$$= \int \frac{d(\sec u + \tan u)}{\sec u + \tan u}$$
$$= \log(\sec u + \tan u) + C.$$

Hence

(XV) $$\int \sec u\,du = \log(\sec u + \tan u) + C.$$

Similarly,

(XVI) $$\int \csc u\,du = \log(\csc u - \cot u) + C.$$

PROBLEMS

1. Integrate:

a. $\int \tan ax\, dx.$

b. $\int \csc \frac{x}{a} dx.$

c. $\int \sec 2t \tan 2t\, dt.$

d. $\int \csc 2y \cot 2y\, dy.$

e. $\int \csc^2 ax\, dx.$

f. $\int \cot \frac{x}{2} dx.$

g. $\int (\sec^2 x^2)\, x\, dx.$

h. $\int \frac{dx}{\sin^2 x}.$

i. $\int (\tan\theta + \cot\theta)^2 d\theta.$

j. $\int \frac{ds}{\cos^2 s}.$

k. $\int (\sec x + \tan x)^2 dx.$

l. $\int \frac{\cos x\, dx}{\sin^2 x}.$

2. Evaluate the following:

a. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \csc^2 x\, dx.$

b. $\int_0^1 \sec^2 \frac{\pi x}{4} dx.$

c. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot 2x\, dx.$

d. $\int_0^2 \tan \frac{\pi x}{6} dx.$

3. The derivative of a certain function is $\sec^2\theta + \tan\theta$, and its value is 5 when $\theta = 0$. Find the function.

4. Plot the branch of the curve $y = \tan\frac{\pi x}{4}$ which passes through the origin, and give the equations of its asymptotes.

5. Find the area bounded by the curve $y = \tan\frac{\pi x}{4}$, the x-axis, and the line $x = 1$.

6. Sketch the branch of the curve $y = \sec\frac{\pi x}{2}$ which crosses the y-axis, and give the equations of its asymptotes.

7. Find the volume generated if the area bounded by the curve $y = \sec\frac{\pi x}{2}$, the x-axis, and the lines $x = \pm\frac{1}{2}$ is revolved about the x-axis.

8. Find the length of the arc of the curve $y = \log \sec x$ from $x = 0$ to $x = \frac{\pi}{3}$.

9. Find the length of the arc of the curve $y = \tan x$ from $x = 0$ to $x = \frac{\pi}{3}$. Use Simpson's rule, taking $n = 6$.

10. Find the volume generated if the area bounded by the curve $y = \cot x$, the x-axis, and the line $x = \frac{\pi}{4}$ is revolved about the x-axis.

199. Parametric Equations. For many purposes curves are represented by parametric equations involving trigonometric functions.* Consider, as an example, the motion of a point on the rim of a flywheel of radius a feet. Taking the origin of coördinates at the center of the wheel, the equation of the path of the point is

$$x^2 + y^2 = a^2.$$

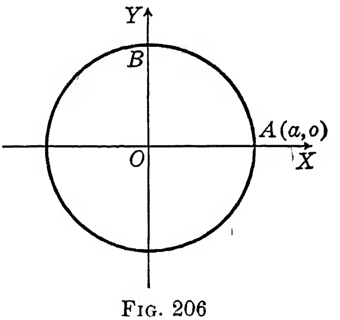


FIG. 206

This equation, however, tells nothing about the motion of the point. In order to describe the motion of the point along its circular path, it is necessary to express x and y as functions of the time. Suppose $x = a \cos \pi t$, $y = a \sin \pi t$, where t denotes the time measured in seconds. These equations show (1) that the moving point starts at A $(a, 0)$ when $t = 0$; (2) that the motion around the circle is counterclockwise; (3) that one revolution is completed in 2 sec. The speed v is given by

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{a^2\pi^2 \sin^2 \pi t + a^2\pi^2 \cos^2 \pi t} = \pi a.$$

* Compare § 90.

Hence the speed is constant, and equal to πa feet per second. The direction in which the point is moving at any instant is found by getting the slope of the curve.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\pi a \cos \pi t}{-\pi a \sin \pi t} = -\cot \pi t.$$

200. The Cycloid. A cycloid is the curve traced by a point on the circumference of a circle which rolls along a straight line. To derive the usual parametric equations

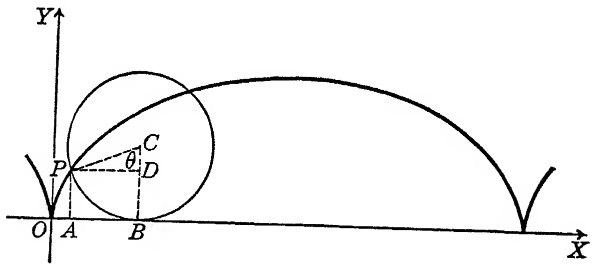


FIG. 207

of the cycloid, we suppose that the circle, of radius a, rolls along the x-axis and that the origin is taken at a point where the curve touches the x-axis. Let $P(x, y)$ represent the position of the point when the circle has turned through an angle θ. Then, in the figure,

$$OB = \text{arc } BP = a\theta,$$

$$x = OA = OB - AB = OB - PD = a\theta - a \sin \theta,$$

$$y = AP = BD = BC - DC = a - a \cos \theta.$$

Hence the parametric equations of the cycloid of the figure are

$$x = a(\theta - \sin \theta), \quad y = a(1 - \cos \theta).$$

EXAMPLE. Find the area of one quadrant of a circle by means of the parametric equations $x = r\cos\theta$, $y = r\sin\theta$.

Solution. The area in question is easily seen to be equal to

$$\int_0^r y\,dx.$$

From the parametric equations, we have

$$y = r\sin\theta \quad\text{and}\quad dx = -r\sin\theta\,d\theta.$$

When $x = 0$, $\cos\theta = 0$ and $\theta = \frac{\pi}{2}$; when $x = r$, $\cos\theta = 1$ and $\theta = 0$. Substituting these values, we find

$$A = \int_0^r y\,dx = -r^2\int_{\frac{\pi}{2}}^0 \sin^2\theta\,d\theta.$$

This integral cannot be evaluated by formulas previously given. It can be integrated by parts (see § 202) or by means of the substitution $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$. Using the latter method, we find that

$$\begin{aligned} A &= -\frac{r^2}{2}\int_{\frac{\pi}{2}}^0 (1-\cos 2\theta)\,d\theta \\ &= -\frac{r^2}{2}\left[\int d\theta - \frac{1}{2}\int \cos 2\theta\cdot 2\,d\theta\right]_{\frac{\pi}{2}}^0 \\ &= -\frac{r^2}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{2}}^0 \\ &= \frac{\pi r^2}{4}. \end{aligned}$$

PROBLEMS

1. Plot the graphs of the following parametric equations:

a. $x = r\sin\theta$, $y = r\cos\theta$. ***c.*** $x = 5\cos\theta$, $y = 3\sin\theta$.
b. $x = 1 + \cos t$, $y = \sin t$. ***d.*** $x = 3\cos\theta$, $y = 5 - 5\sin\theta$.
e. $x = \sec\theta$, $y = \tan\theta$.

2. Eliminate the parameter in each case of Problem 1 and name the locus.

3. Prove that $x = a + r \cos \theta$, $y = b + r \sin \theta$, are parametric equations of a circle, and find the center and radius.

4. Write parametric equations for the motion of a point on the rim of a flywheel making one revolution per second; 180 revolutions per minute.

5. Find the inclination of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$, at the point where $\theta = \frac{1}{2}\pi$.

6. Find an expression for the slope of the ellipse $x = a \cos \theta$, $y = b \sin \theta$, in terms of θ.

7. Plot the graph of $x = a \cos^3 \theta$, $y = a \sin^3 \theta$.

8. Find the slope of the curve in Problem 7 at the point where $\theta = 30°$.

9. The equations of motion of a point are $x = 10 \sin t$, $y = 5 \cos t$. Find the speed and direction of motion when $t = 0$, $\frac{1}{4}\pi$, $\frac{1}{2}\pi$.

10. A wheel of radius 3 ft. rotates at a constant angular velocity of 2 rad./sec. Using the parametric equations $x = 3 \cos \theta$, $y = 3 \sin \theta$, find the x- and y-components of the velocity of a point on the rim. Find also the resultant velocity and direction of motion of the point when $\theta = 30°$.

11. A point moves in such a way that $x = 10(\cos t + t \sin t)$, $y = 10(\sin t - t \cos t)$. Plot the path from $t = 0$ to $t = 8$, taking t at intervals of 1 rad.

12. In Problem 11 let t denote time in seconds and find v_x, v_y, and v for $t = 2$.

13. Write parametric equations of the motion of a point on the rim of a wheel of radius 2 ft. which rolls along the ground, making one revolution in 2 sec.

14. What is the speed of the center of the wheel of Problem 13? When the point on the rim is at the top of the wheel, show that it is moving twice as fast as the center.

15. Find the speed of the point on the rim of the wheel of Problem 13 when it is on the same level as the center of the wheel.

16. Using the parametric equations, find the length of one quadrant of a circle.

17. Find the area of the ellipse $x = a \cos \theta$, $y = b \sin \theta$.

18. Find the area of one arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$.

19. Find the length of one arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$.

HINT. Simplify the integral by using the formula for $\sin \frac{\theta}{2}$.

201. Powers and Products of Trigonometric Functions. Expressions of the form $\sin^m u \cos^n u\, du$, $\tan^m u \sec^n u\, du$, and $\cot^m u \csc^n u\, du$ are frequently integrable by means of the power formula. The method is best explained by illustrating it.

EXAMPLE 1. Find the value of $\int \sin u \cos^3 u\, du$.

Set $v = \cos u$; then $dv = - \sin u\, du$. Hence we have

$$\int \sin u \cos^3 u\, du = - \int v^3 dv$$
$$= - \frac{v^4}{4} + C = - \frac{1}{4} \cos^4 u + C.$$

EXAMPLE 2. Find the value of $\int \cos^3 u\, du$.

If we set $v = \cos u$, $dv = - \sin u\, du$. But no power of $\sin u$ occurs. On the other hand, $\cos u\, du = d \sin u$; and, if this is factored out, the other factor, $\cos^2 u$, may be transformed into powers of $\sin u$ by the formula $\sin^2 u + \cos^2 u = 1$. Hence we have

$$\int \cos^3 u\, du = \int (1 - \sin^2 u) \cos u\, du$$
$$= \int \cos u\, du - \int \sin^2 u \cos u\, du$$
$$= \sin u - \int \sin^2 u \cos u\, du.$$

In this last integral, set $v = \sin u$; whence $dv = \cos u\,du$, and

$$\int \sin^2 u \cos u\,du = \frac{\sin^3 u}{3} + C.$$

Therefore $$\int \cos^3 u\,du = \sin u - \frac{\sin^3 u}{3} + C.$$

EXAMPLE 3. Find the value of $\int \tan^4 u \sec^4 u\,du$.

Set $v = \tan u$; then $dv = \sec^2 u\,du$. Hence we have

$$\begin{aligned}\int \tan^4 u \sec^4 u\,du &= \int \tan^4 u \sec^2 u \sec^2 u\,du \\ &= \int \tan^4 u (1 + \tan^2 u) \sec^2 u\,du \\ &= \int \tan^4 u \sec^2 u\,du + \int \tan^6 u \sec^2 u\,du \\ &= \frac{\tan^5 u}{5} + \frac{\tan^7 u}{7} + C.\end{aligned}$$

The essence of this method is to set v equal to one of the trigonometric functions and to factor out its differential. The other factor must then be reducible to a polynomial involving only integral powers of the trigonometric function chosen. In order to use it successfully, one must be thoroughly familiar with the differentials of the trigonometric functions and the three formulas from trigonometry which permit transferring from one function into another. These are given below, for convenience:

$$\begin{aligned} d \sin u &= \cos u\,du, & d \cos u &= -\sin u\,du, \\ d \tan u &= \sec^2 u\,du, & d \cot u &= -\csc^2 u\,du, \\ d \sec u &= \sec u \tan u\,du, & d \csc u &= -\csc u \cot u\,du. \end{aligned}$$

$$\sin^2 u + \cos^2 u = 1,$$
$$1 + \tan^2 u = \sec^2 u,$$
$$1 + \cot^2 u = \csc^2 u.$$

Since these formulas involve squares of the functions, the method is bound to fail in the following cases: $\sin^m u \cos^n u\,du$, when both m and n are even; $\tan^m u \sec^n u\,du$ and $\cot^m u \csc^n u\,du$, when m is even and n is odd. In these cases the substitutions can be made, but the resulting expression involves radicals. For example, take $\int \cot^2 u \csc^3 u\,du$. If we set $v = \cot u$, $dv = -\csc^2 u\,du$, and the result is

$$\int \cot^2 u \csc^3 u\,du = \int \cot^2 u \sqrt{1 + \cot^2 u}\,(-\csc^2 u\,du).$$

We get no better results if we try $v = \csc u$.

PROBLEMS

1. Integrate

a. $\int \sin^2 x \cos x\,dx.$

b. $\int \sin x \cos x\,dx.$

c. $\int \cos^2 x \sin x\,dx.$

d. $\int \cos^3 2\,\theta \sin 2\,\theta d\theta.$

e. $\int \dfrac{\sin^3 x\,dx}{\cos^2 x}.$

f. $\int \tan^3 x\,dx.$

g. $\int \cot^2 x\,dx.$

h. $\int \cot^3 x\,dx.$

i. $\int \cot^3 x \csc^4 x\,dx.$

j. $\int (\tan t + \cot t)^3 dt.$

k. $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^3 x\,dx.$

l. $\int_0^1 \cos^2 \dfrac{\pi x}{2} \sin \dfrac{\pi x}{2}\,dx.$

2. Find the volume of the ellipsoid generated by rotating the curve $x = a \cos\theta$, $y = b \sin\theta$, about (*a*) the x-axis; (*b*) the y-axis.

3. Find the area included within the hypocycloid $x = a\cos^3\theta$, $y = a\sin^3\theta$.

4. Find the volume generated by rotating the curve in Problem 3 about either axis.

202. Integration by Parts. The formula for differentiating the product of two functions is

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx},$$

or, in differential notation,

$$d(uv) = u\,dv + v\,du;$$

whence

$$u\,dv = d(uv) - v\,du.$$

Integrating both sides of this equation, we have

$$\text{(XVII)} \qquad \int u\,dv = uv - \int v\,du.$$

This very important formula is known as the formula for *integration by parts*. Its use will be illustrated by some examples.

EXAMPLE 1. Integrate $\int x \sin x\,dx$.

Solution. Comparing this integral with the formula, we have $u\,dv = x \sin x\,dx$. This is the only relation which will guide us in the choice of the functions u and v. There are two possibilities:

1. If $u = x$, then $dv = \sin x\,dx$.
2. If $u = \sin x$, then $dv = x\,dx$.

Since the proper choice can be made only by experience, we will try each of the possibilities.

1. If $u = x$, we have $du = dx$, and if $dv = \sin x\,dx$, we have $v = -\cos x + C_1$. Substitution in the formula gives

$$\begin{aligned}\int x \sin x\,dx &= x(-\cos x + C_1) - \int(-\cos x + C_1)\,dx\\ &= -x\cos x + C_1 x + \sin x - C_1 x + C\\ &= \sin x - x\cos x + C.\end{aligned}$$

It will be observed that the constant C_1 cancels out of the final result. As this always happens, it is not necessary to add an arbitrary constant when integrating dv to find v.

2. If $u = \sin x$, $du = \cos x\,dx$, and if $dv = x\,dx$, $v = \frac{x^2}{2}\cdot$ Substitution in the formula gives

$$\int x \sin x\,dx = \frac{x^2}{2}\sin x - \int \frac{x^2}{2}\cos x\,dx.$$

Since the integral on the right-hand side is more difficult than the original problem, the second choice of u and dv is not good.

Example 2. Integrate $\int x^2 e^x\,dx$.

Solution. Let $$u = x^2.$$

Then $$dv = e^x\,dx,$$

and $$du = 2\,x\,dx, \quad v = e^x.$$

Substitution in the formula gives

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx. \tag{1}$$

To find $\int x e^x\,dx$, we again use the method of integration by parts.

Let $$u = x.$$

Then $$dv = e^x\,dx,$$

and $$du = dx, \quad v = e^x.$$

Substitution in the formula gives

$$\begin{aligned}\int x e^x dx &= x e^x - \int e^x\,dx \\ &= x e^x - e^x.\end{aligned}$$

Substitution of this value in equation (1) gives the final result

$$\begin{aligned}\int x^2 e^x dx &= x^2 e^x - 2\,x e^x + 2\,e^x + C \\ &= e^x(x^2 - 2\,x + 2) + C.\end{aligned}$$

EXAMPLE 3. Find $\int \sin^2 x\, dx$.

Solution. Let $u = \sin x$ and $dv = \sin x\, dx$.

Then $du = \cos x\, dx$ and $v = -\cos x$.

Hence
$$\int \sin^2 x\, dx = -\sin x \cos x + \int \cos^2 x\, dx$$
$$= -\sin x \cos x + \int (1 - \sin^2 x)\, dx$$
$$= -\sin x \cos x + x - \int \sin^2 x\, dx.$$

Transposing,
$$2\int \sin^2 x\, dx = x - \sin x \cos x,$$
$$\int \sin^2 x\, dx = \frac{x}{2} - \frac{1}{2} \sin x \cos x + C.$$

EXAMPLE 4. Find $\int \sec^3 x\, dx$.

Solution. Let $u = \sec x$ and $dv = \sec^2 x\, dx$.

Then $du = \sec x \tan x\, dx$ and $v = \tan x$.

Hence
$$\int \sec^3 x\, dx = \sec x \tan x - \int \sec x \tan^2 x\, dx.$$

As in the previous example, we transform the integral on the right so as to get the unknown integral as one term. The proper substitution is
$$\tan^2 x = \sec^2 x - 1,$$
which gives
$$\int \sec^3 x\, dx = \sec x \tan x - \int \sec x (\sec^2 x - 1)\, dx$$
$$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$
$$= \sec x \tan x + \log(\sec x + \tan x) - \int \sec^3 x\, dx.$$

Transposing and dividing by 2, we have
$$\int \sec^3 x\, dx = \tfrac{1}{2} \sec x \tan x + \tfrac{1}{2} \log(\sec x + \tan x) + C.$$

PROBLEMS

1. Integrate

a. $\int x \cos x \, dx.$	*e.* $\int x \tan^2 x \, dx.$	*i.* $\int x^2 \sin x \, dx.$
b. $\int x e^{3x} dx.$	*f.* $\int x \log x \, dx.$	*j.* $\int x^2 \cos 2x \, dx$
c. $\int x e^{-2x} dx.$	*g.* $\int \log x \, dx.$	*k.* $\int x^2 e^{-x} dx.$
d. $\int x \sin 2x \, dx.$	*h.* $\int x^n \log x \, dx.$	*l.* $\int \tan^3 x \, dx.$

2. Find the area bounded by the curve $y = \log x$, the x-axis, and the line $x = 10$.

3. Find the coördinates of the centroid of the area bounded by the arch of the curve $y = \sin x$ between $x = 0$ and $x = \pi$ and by the x-axis.

4. Find the area bounded by the curve $y = xe^x$, the x-axis and the line $x = 4$.

5. Find the volume generated by revolving the area of Problem 4 about the x-axis.

203. The Trigonometric Substitutions. Many definite integrals involve one of the three radicals $\sqrt{a^2 - u^2}$, $\sqrt{u^2 - a^2}$, and $\sqrt{a^2 + u^2}$, where a is some constant and u is a function of x. Expressions of this kind can be freed of radicals by a proper trigonometrical substitution. These substitutions are as follows:

To rationalize $\sqrt{a^2 - u^2}$, substitute $u = a \sin z$.
To rationalize $\sqrt{u^2 - a^2}$, substitute $u = a \sec z$.
To rationalize $\sqrt{a^2 + u^2}$, substitute $u = a \tan z$.

For, in the first case,

$$\sqrt{a^2 - u^2} = \sqrt{a^2 - a^2 \sin^2 z} = \sqrt{a^2(1 - \sin^2 z)} = \sqrt{a^2 \cos^2 z} = a \cos z.$$

The other statements are verified in like manner.

EXAMPLE 1. Find in terms of x the value of $\int \frac{\sqrt{4x^2-9}}{x}\,dx$.

Solution. Here $u = 2x$ and $a = 3$. We therefore substitute $2x = 3\sec z$, whence $x = \frac{3}{2}\sec z$ and $dx = \frac{3}{2}\sec z \tan z\,dz$. Therefore

$$\int \frac{\sqrt{4x^2-9}}{x}\,dx = \int \frac{\sqrt{9\sec^2 z - 9}}{\frac{3}{2}\sec z}\cdot\frac{3}{2}\sec z\tan z\,dz$$

$$= 3\int \tan^2 z\,dz$$

$$= 3\int (\sec^2 z - 1)\,dz$$

$$= 3\tan z - 3z + C.$$

In order to get the result in terms of x, proceed as follows: Since $2x = 3\sec z$, $\sec z = \frac{2x}{3}$. Draw a right triangle whose acute angle is z. Since $\sec z = \frac{2x}{3}$, the hypotenuse will be $2x$, and the adjacent side will be 3. The opposite side is then $\sqrt{4x^2-9}$ and $\tan z = \frac{\sqrt{4x^2-9}}{3}$. Since $\sec z = \frac{2x}{3}$, z is the angle of which the secant is $\frac{2x}{3}$, written either $\sec^{-1}\left(\frac{2x}{3}\right)$ or arc sec $\frac{2x}{3}$. The latter symbol is preferable. Substituting these above, we have

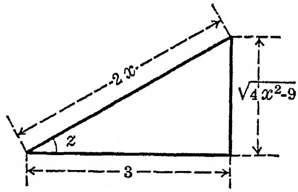


FIG. 208

$$\int \frac{\sqrt{4x^2-9}}{x}\,dx = \sqrt{4x^2-9} - 3\,\text{arc sec}\,\frac{2x}{3} + C.$$

When the integral is a definite integral, it is usually better to change the limits of integration when the substitution is made.

EXAMPLE 2. Find the area of one quadrant of the circle $x^2 + y^2 = r^2$.

Solution. The required area is obviously

$$A = \int_0^r y\,dx$$

$$= \int_0^r \sqrt{r^2 - x^2}\,dx.$$

Set $x = r \sin z$; then $dx = r \cos z\,dz$. When $x = r$, $\sin z = 1$ or $z = \frac{\pi}{2}$; when $x = 0$, $\sin z = 0$ or $z = 0$.

Therefore

$$A = \int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2 \sin^2 z}\, r \cos z\,dz$$

$$= r^2 \int_0^{\frac{\pi}{2}} \cos^2 z\,dz$$

$$= r^2 \left[\frac{z}{2} + \frac{1}{2} \sin z \cos z\right]_0^{\frac{\pi}{2}} \text{ (by parts)}$$

$$= \frac{\pi r^2}{4},$$

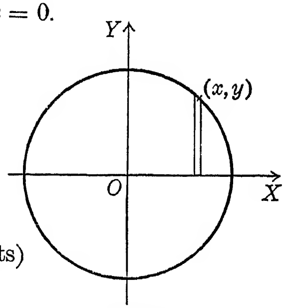


FIG. 209

which agrees with the well-known formula for the area of a circle.

PROBLEMS

1. Integrate

a. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}}$.

b. $\displaystyle\int \frac{dx}{\sqrt{x^2 + a^2}}$.

c. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}}$.

d. $\displaystyle\int \sqrt{a^2 - x^2}\,dx$.

e. $\displaystyle\int \sqrt{x^2 + a^2}\,dx$.

f. $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4 - x^2}}$.

g. $\displaystyle\int_0^3 x^3\sqrt{9 - x^2}\,dx$.

h. $\displaystyle\int_1^2 \frac{dx}{x^2\sqrt{5 - x^2}}$.

i. $\displaystyle\int_0^4 \frac{dx}{(x^2 + 9)^{\frac{3}{2}}}$.

2. Find the area of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

3. Find the area under one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

4. Find the volume generated by rotating the area in Problem 3 about the x-axis.

5. Find the length of the arc of the parabola $y^2 = 2px$ between the points for which $y = 0$ and $y = p$.

6. Find the volume generated by revolving one arch of the sine curve about the x-axis.

7. Find the area bounded by the hyperbola $x^2 - y^2 = a^2$ and the line $x = 2a$.

8. Find the area bounded by the curve $9y^2 = 4(1 + x^2)^3$ and the lines $y = 0$, $x = 0$, and $x = 1$.

9. A horizontal cylindrical tank of radius 4 ft. is full of water. Find the pressure on each end.

10. Obtain a formula for the area of a segment of a circle of radius r units cut off by a chord distant a units from the center.

11. Find by integration the volume of the solid in Problem 19, page 330.

12. Find by integration the values of the following definite integrals:

a. Page 311, Problem 3. *b.* Page 313, Problem 2 a.

COLLECTION OF FORMULAS

GREEK ALPHABET

Letters		Names	Letters		Names	Letters		Names
A	α	Alpha	I	ι	Iota	P	ρ	Rho
B	β	Beta	K	κ	Kappa	Σ	σ ς	Sigma
Γ	γ	Gamma	Λ	λ	Lambda	T	τ	Tau
Δ	δ	Delta	M	μ	Mu	Υ	υ	Upsilon
E	ϵ	Epsilon	N	ν	Nu	Φ	ϕ	Phi
Z	ζ	Zeta	Ξ	ξ	Xi	X	χ	Chi
H	η	Eta	O	o	Omicron	Ψ	ψ	Psi
Θ	θ	Theta	Π	π	Pi	Ω	ω	Omega

FORMULAS FROM ALGEBRA

1. Binomial theorem (n being a positive integer):

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{\underline{|2}}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{\underline{|3}}a^{n-3}b^3 + \cdots$$
$$+ \frac{n(n-1)(n-2)\cdots(n-r+2)}{\underline{|r-1}}a^{n-r+1}b^{r-1} + \cdots.$$

2. $n! = \underline{|n} = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1)n.$

3. The solution of the quadratic equation $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

4. When a quadratic equation is reduced to the form $x^2 + px + q = 0$,

$p =$ sum of roots with sign changed, and $q =$ product of roots.

5. In an arithmetical series,

$$l = a + (n-1)d;\ s = \frac{n}{2}(a + l) = \frac{n}{2}[2a + (n-1)d].$$

6. In a geometrical series,

$$l = ar^{n-1};\ s = \frac{rl - a}{r - 1} = \frac{a(r^n - 1)}{r - 1}.$$

FORMULAS FROM GEOMETRY

In formulas 7–20, r denotes radius, h altitude, B area of base, s slant height, and α an angle measured in radians.

7. Circumference of circle $= 2\pi r$.

8. Arc of sector $= r\alpha$.

9. Area of circle $= \pi r^2$.

10. Area of sector $= \frac{1}{2} r^2\alpha$.

11. Volume of prism $= Bh$.

12. Volume of pyramid $= \frac{1}{3} Bh$.

13. Volume of frustum of pyramid $= \frac{1}{3} h\left(B_1 + B_2 + \sqrt{B_1 B_2}\right)$.

14. Volume of right circular cylinder $= \pi r^2 h$.

15. Lateral surface of right circular cylinder $= 2\pi rh$.

16. Volume of right circular cone $= \frac{1}{3}\pi r^2 h$.

17. Lateral surface of right circular cone $= \pi rs$.

18. Volume of frustum of right circular cone $= \frac{1}{3}\pi h\left(r_1^2 + r_2^2 + r_1 r_2\right)$.

19. Volume of sphere $= \frac{4}{3}\pi r^3$.

20. Surface of sphere $= 4\pi r^2$.

FORMULAS FROM TRIGONOMETRY

21. $\cot x = \dfrac{1}{\tan x};\ \sec x = \dfrac{1}{\cos x};\ \csc x = \dfrac{1}{\sin x}.$

22. $\tan x = \dfrac{\sin x}{\cos x};\ \cot x = \dfrac{\cos x}{\sin x}.$

23. $\sin^2 x + \cos^2 x = 1;\ 1 + \tan^2 x = \sec^2 x;\ 1 + \cot^2 x = \csc^2 x.$

24. $\sin(\pi - x) = \sin x;\ \sin(\pi + x) = -\sin x;$
$\cos(\pi - x) = -\cos x;\ \cos(\pi + x) = -\cos x;$
$\tan(\pi - x) = -\tan x;\ \tan(\pi + x) = \tan x.$

25. $\sin\left(\frac{\pi}{2}-x\right)=\cos x$; $\sin\left(\frac{\pi}{2}+x\right)=\cos x$;

$\cos\left(\frac{\pi}{2}-x\right)=\sin x$; $\cos\left(\frac{\pi}{2}+x\right)=-\sin x$;

$\tan\left(\frac{\pi}{2}-x\right)=\cot x$; $\tan\left(\frac{\pi}{2}+x\right)=-\cot x$.

26. $\sin(x+y)=\sin x\cos y+\cos x\sin y$.

27. $\sin(x-y)=\sin x\cos y-\cos x\sin y$.

28. $\cos(x+y)=\cos x\cos y-\sin x\sin y$.

29. $\cos(x-y)=\cos x\cos y+\sin x\sin y$.

30. $\tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}$. **31.** $\tan(x-y)=\dfrac{\tan x-\tan y}{1+\tan x\tan y}$.

32. $\sin 2x=2\sin x\cos x$; $\cos 2x=\cos^2 x-\sin^2 x$; $\tan 2x=\dfrac{2\tan x}{1-\tan^2 x}$.

33. $\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}$; $\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}$; $\tan x=\dfrac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$.

34. $\cos^2 x=\frac{1}{2}+\frac{1}{2}\cos 2x$; $\sin^2 x=\frac{1}{2}-\frac{1}{2}\cos 2x$.

35. $1+\cos x=2\cos^2\frac{x}{2}$; $1-\cos x=2\sin^2\frac{x}{2}$.

36. $\sin\frac{x}{2}=\pm\sqrt{\dfrac{1-\cos x}{2}}$; $\cos\frac{x}{2}=\pm\sqrt{\dfrac{1+\cos x}{2}}$;

$\tan\frac{x}{2}=\pm\sqrt{\dfrac{1-\cos x}{1+\cos x}}$.

37. $\sin x+\sin y=2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$.

38. $\sin x-\sin y=2\cos\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)$.

39. $\cos x+\cos y=2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$.

40. $\cos x-\cos y=-2\sin\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)$.

41. $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$; law of sines.

42. $a^2=b^2+c^2-2bc\cos A$; law of cosines.

43. Area of triangle $=\frac{1}{2}bc\sin A$

$$=\frac{a^2\sin B\sin C}{2\sin(B+C)}$$

$$=\sqrt{s(s-a)(s-b)(s-c)}.$$

$$s=\tfrac{1}{2}(a+b+c).$$

FORMULAS FROM ANALYTIC GEOMETRY AND CALCULUS

44. Distance between two points, p. 9.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

45. Slope of line joining two points, p. 17.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

46. Mid-point formulas, p. 9.

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}$$

47. Point dividing a segment in a given ratio, p. 13.

$$x = \frac{x_1 + rx_2}{1 + r}, \quad y = \frac{y_1 + ry_2}{1 + r}$$

48. Angle between two lines, pp. 20, 21.

Condition for parallelism: $m_1 = m_2$

Condition for perpendicularity: $m_1 m_2 = -1$

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

49. Equation of a straight line, pp. 26, 28.

Point-slope form: $y - y_1 = m(x - x_1)$

Slope-intercept form: $y = mx + b$

Two-point form: $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$

Intercept form: $\dfrac{x}{a} + \dfrac{y}{b} = 1$

50. Distance from a line to a point, p. 41.

$$d = \frac{y_1 - mx_1 - b}{\sqrt{m^2 + 1}}$$

51. Equations of curves:

Circle with center (h, k) and radius r, p. 195.

$$(x - h)^2 + (y - k)^2 = r^2$$

Parabola with vertex at the origin and with focus on the x-axis, p. 210.

$$y^2 = 2px$$

Ellipse with center at the origin and with foci on the x-axis, p. 223,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Hyperbola with center at the origin and with foci on the x-axis, p. 230.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Equilateral hyperbola with center at the origin and with the coördinate axes for asymptotes, p. 236.

$$xy=C$$

Other curves with historical names:

Catenary: $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$

Cissoid: $y^2(2a-x)=x^3$

Conchoid: $x^2y^2=(y+a)^2(b^2-y^2)$

Cubical parabola: $y=ax^3$

Cycloid: $x=a(\theta-\sin\theta),\quad y=a(1-\cos\theta)$

Folium: $x^3+y^3-3axy=0$

Hypocycloid of four cusps: $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$

Lemniscate: $(x^2+y^2)^2=a^2(x^2-y^2)$

Parabola: $\sqrt{x}+\sqrt{y}=\sqrt{a}$

Probability curve: $y=e^{-x^2}$

Semicubical parabola: $y^2=ax^3$

Strophoid: $(a-x)y^2=(a+x)x^2$

Witch: $y=\frac{8a^3}{x^2+4a^2}$

52. Radius of curvature, p. 263:

$$R=\pm\frac{(1+y'^2)^{\frac{3}{2}}}{y''}$$

53. Formulas for differentiation, pp. 115, 367, 372, 393, 401.

(I) $$\frac{d}{dx}x^n=nx^{n-1}.$$

(I a) $$\frac{d}{dx}x=1.$$

(II) $$\frac{d}{dx}c=0.$$

(III) $$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$$

(IV) $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

(IV a) $$\frac{d}{dx}(cv) = c\frac{dv}{dx}.$$

(V) $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

(VI) $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

(VII) $$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

(VIII) $$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}.$$

(IX) $$\frac{d}{dx}\log u = \frac{1}{u}\frac{du}{dx}.$$

(IX a) $$\frac{d}{dx}\log x = \frac{1}{x}.$$

(IX b) $$\frac{d}{dx}\log_a u = \frac{1}{u}\frac{du}{dx}\log_a e.$$

(IX c) $$\frac{d}{dx}\log_{10} u = \frac{0.434}{u}\frac{du}{dx}.$$

(X) $$\frac{d}{dx}a^u = a^u \log a\frac{du}{dx}.$$

(X a) $$\frac{d}{dx}a^x = a^x \log a.$$

(X b) $$\frac{d}{dx}e^u = e^u\frac{du}{dx}.$$

(X c) $$\frac{d}{dx}e^x = e^x.$$

(XI) $$\frac{d}{dx}\sin u = \cos u\frac{du}{dx}.$$

(XII) $$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}.$$

(XIII) $$\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}.$$

(XIV) $$\frac{d}{dx}\cot u = -\csc^2 u \frac{du}{dx}.$$

(XV) $$\frac{d}{dx}\sec u = \sec u \tan u \frac{du}{dx}.$$

(XVI) $$\frac{d}{dx}\csc u = -\csc u \cot u \frac{du}{dx}.$$

54. Formulas for integration, pp. 269, 379, 404.

(I) $$\int du = u + C.$$

(II) $$\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx.$$

(III) $$\int a\, du = a \int du.$$

(IV) $$\int u^n du = \frac{u^{n+1}}{n+1} + C, \text{ if } n \neq -1.$$

(V) $$\int \frac{du}{u} = \log u + C.$$

(VI) $$\int e^u du = e^u + C.$$

(VI a) $$\int a^u du = \frac{a^u}{\log a} + C.$$

(VII) $$\int \sin u\, du = -\cos u + C.$$

(VIII) $$\int \cos u\, du = \sin u + C.$$

(IX) $$\int \sec^2 u\, du = \tan u + C.$$

(X) $$\int \csc^2 u\, du = -\cot u + C.$$

(XI) $$\int \sec u \tan u\, du = \sec u + C.$$

(XII) $$\int \csc u \cot u\, du = -\csc u + C.$$

(XIII) $$\int \tan u\, du = \log \sec u + C.$$

(XIV) $$\int \cot u\, du = \log \sin u + C.$$

(XV) $$\int \sec u\, du = \log(\sec u + \tan u) + C.$$

(XVI) $$\int \csc u\, du = \log(\csc u - \cot u) + C.$$

55. Approximate evaluation of $I = \int_a^b f(x)\,dx$:

The interval $b - a$ is divided into n parts, each equal to Δx. Let the abscissas of the points of division be

$$x_0 = a,\ x_1,\ x_2,\ \cdots x_n = b.$$

The corresponding values of $f(x)$ are

$$y_0 = f(x_0),\ y_1 = f(x_1) \cdots y_n = f(x_n).$$

Trapezoidal rule (any number of parts):

$$I = (\tfrac{1}{2} y_0 + y_1 + \cdots + y_{n-1} + \tfrac{1}{2} y_n)\,\Delta x.$$

Simpson's (parabolic) rule (even number of parts, n even):

$$I = (y_0 + 4\,y_1 + 2\,y_2 + 4\,y_3 + 2\,y_4 + \cdots + 4\,y_{n-1} + y_n)\frac{\Delta x}{3}.$$

ANSWERS

NOTE. In calculating the following tables of answers, results involving exponentials and logarithms were calculated with the aid of four-place tables. Angles expressed in decimals were obtained by means of four-place tables; those expressed in degrees and minutes were obtained with five-place tables. If tables correct to more or fewer places of decimals are used, slightly different results may be expected.

Pages 4, 5

7. *a.* 32. *b.* 30. *c.* 24. *d.* 32.

Pages 14–16

9. *a.* 26.32.

Pages 23–25

4. $56.01°$ ($56° 1'$). **5.** $73.20°$ ($73° 12'$). **6.** 0.6602 or -16.67.

10. *a.* $49.76°$ ($49° 46'$), $55.31°$ ($55° 18'$), $74.93°$ ($74° 56'$).

b. $35.84°$ ($35° 50'$), $49.76°$ ($49° 46'$), $94.40°$ ($94° 24'$).

c. $48.17°$ ($48° 11'$), $55.63°$ ($55° 37'$), $76.20°$ ($76° 12'$).

d. $22.38°$ ($22° 22'$), $37.88°$ ($37° 53'$), $119.74°$ ($119° 45'$).

11. *b.* $78.69°$ ($78° 41'$), $101.31°$ ($101° 19'$). *c.* $60.47°$ ($60° 28'$).

12. *b.* $45°$, $90°$, $90°$, $135°$. *c.* $71.57°$ ($71° 34'$).

13. *b.* $45°$, $135°$. *c.* $36.87°$ ($36° 52'$).

Pages 32, 33

2. i. $122.47°$ ($122° 28'$). iii. $119.74°$ ($119° 45'$). v. $2.73°$ ($2° 44'$).

ii. $36.87°$ ($36° 52'$). iv. $85.60°$ ($85° 36'$). vi. $82.88°$ ($82° 53'$).

10. *c.* $71.57°$ ($71° 34'$), $41.63°$ ($41° 38'$), $66.80°$ ($66° 48'$).

12. $x - 4y + 26 = 0$, $x - 4y - 8 = 0$, $4x + y - 32 = 0$, $4x + y + 2 = 0$.

13. $3x + 2y = 24$ or $27x + 2y = 72$.

Pages 35, 36

6. $C = (3, -2)$, $R = 5$.

8. $C = (\frac{27}{14}, -\frac{59}{14})$, $R = 7.27$.

9. $(-\frac{80}{47}, \frac{205}{188})$.

11. Length $= \frac{6}{5}\sqrt{10}$.

12. Length $= 2\sqrt{5}$.

13. Area $= 5$.

14. Area $= 20$.

18. $C = (43.36, 44.90)$, area $= 2245$.

19. $C = (18.21, 10.47)$, $BC = 10.69$.

Pages 45, 46

4. *a.* 26. *b.* 37. *c.* 35. *d.* $\frac{75}{2}$.
5. 134.5.
13. $3x + 6y - 3 = 0$ and $3x - 14y - 3 = 0$.
14. $C = (2, 4)$, $R = 2\sqrt{5}$.

Pages 47, 48

1. *c.* 15. *e.* 45°, 26.56° (26° 34′), 108.44° (108° 26′). *h.* $C = (3, 0)$, $R = 5$.
4. *c.* 30. *d.* 90°, 53.13° (53° 8′), 36.87° (36° 52′).
   e. $y - 3 = 0$, $3x + y - 15 = 0$, $x - y - 1 = 0$.
   f. $C = (4, 3)$, $R = \sqrt{5}$.
   g. $x + 2y - 10 = 0$, $2x + y - 11 = 0$, $2x - y - 5 = 0$.
   h. (2, 4), (5, 1), (5, 5).
7. *a.* 30.51° (30° 31′), 59.49° (59° 29′), 90°.
   b. $x + 8y - 9 = 0$, $7x - 9y + 2 = 0$, $7x + y - 8 = 0$.
   c. $C = (1, 1)$, $R = 2$.
   d. $3x + 4y - 7 = 0$, $4x - 3y - 1 = 0$, $12x + 5y - 17 = 0$.
   e. $(-\frac{3}{5}, \frac{11}{5})$, $(\frac{11}{5}, \frac{13}{5})$, $(\frac{23}{13}, -\frac{11}{13})$.

Pages 60–62

8. $\frac{1}{2}$.
9. $a = \sqrt{28}$, $b = \sqrt{\frac{28}{3}}$.
10. $a = 5$, $b = 3$, $c = -1$.
16. $7x^2 + 14xy - 7y^2 - 12x + 36y - 36 = 0$.

Pages 64–66

1. *a.* $26x - 16y = 59$. *b.* $x = y$.
2. *a.* $y^2 + 44x - 220 = 0$. *b.* $x^2 - 6y + 39 = 0$.
3. *a.* $y^2 + 48x - 288 = 0$. *b.* $x^2 - 10y + 55 = 0$.
4. $xy = 16$.
5. *a.* $x^2 + y^2 - 8x + 10y - 59 = 0$.
   e. $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$.
6. $x^2 + y^2 - 8x = 0$.
7. $x^2 + y^2 - 9x - 9y + 36 = 0$.
8. $x^2 - 6y - 9 = 0$.
9. $x^2 - y^2 - 2 = 0$.
10. $x^2 + y^2 - 6x + 4 = 0$.
11. *a.* $20x = k$. *b.* $x^2 + y^2 = \frac{k}{2} - 25$. *c.* $x = \frac{5(k+1)}{k-1}$.
12. $x^2 + y^2 = 100$.
13. *a.* $3x^2 - y^2 \pm 8x - 16 = 0$.
    b. $x^2 + y^2 \pm 8y = 16$.
    e. $9x^2 + 25y^2 = 225$.
14. *a.* $x^2 - 3x - 4y + 2 = 0$.
    b. $xy - x - 14y + 46 = 0$.
    c. $xy - 4x - 2y + 4 = 0$.
15. *a.* $x = 3$.
    b. $x^2 - 6x + 18y - 72 = 0$.
    c. $x^2 + y^2 = 36$.

Pages 67, 68

1. $(2, 4), (0.5, -2)$.
2. $(4, -5)$.
4. $(4, -9), (-4, -9)$.
11. $(2, 1), (-2, 1)$.
13. $(0, 0), (1.73, 0.43), (-1.73, -0.43)$.
15. $(6, -4), (-1, -5)$.
7. $(3.45, 2.03), (-3.45, -2.03), (2.03, 3.45), (-2.03, -3.45)$.
16. 15.748.
17. 2.121.
18. $(1, 1), (-\frac{15}{17}, \frac{145}{17})$.
19. $m^2 < 20$.

Page 69

5. $\frac{48}{5}$.
8. $16x^2 + 25y^2 = 400$.
9. $9x^2 - 16y^2 = 144$.
11. $x^2 + y^2 \pm 10y - 25 = 0$.

Pages 74, 75

1. *d.* $y = \pm x\sqrt{x^2 - 2},\ x = \pm\sqrt{1 \pm \sqrt{y^2 + 1}}$.
8. *a.* -0.11. *b.* -0.158. *c.* $-\frac{8}{15}$. *d.* 4.62. *e.* 0.301. *f.* -0.04.
9. *b.* $\Delta y = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$.
   d. $\Delta y = \dfrac{-36\,\Delta x}{x^2 + x\Delta x}$.
   f. $\Delta y = (2ax + b)\Delta x + a(\Delta x)^2$.

Pages 82, 83

4. $\Delta y = 0.125$. 6. *a.* $-4, -3.5$. *c.* $0, -0.5$. 7. *a.* 3. *c.* 0. *e.* $-\frac{3}{4}$. *g.* $-\frac{3}{4}$.
8. *a.* $71.57°$ $(71°\,34')$. *c.* $0°$. *e.* $143.13°$ $(143°\,8')$. *g.* $143.13°$ $(143°\,8')$.
9. *a.* $-3, 3$. *b.* $3, 0, 3$. *e.* $m = \dfrac{12}{x^2}$. *g.* $-8, 0, 8$. *i.* $\frac{2}{3}, \frac{3}{2}, 6$. *j.* $m = 3x^2 - 2x$. *k.* $m = \dfrac{-4}{(x+2)^2}$. *l.* $-4, -2, 0, 2, 4$. *n.* $-6, -2, 0, 2, 6$. *p.* $m = 3x^2 - 3$. *r.* $m = -\dfrac{2}{x^3}$.

Pages 91, 92

1. *a.* $8x$. *c.* $3x^2 - 1$. *d.* $-\dfrac{8}{x^2}$. *f.* $\dfrac{-4}{(x-2)^2}$. *g.* $1 - \dfrac{1}{x^2}$. *h.* $-\dfrac{2}{x^3}$.
4. $\dfrac{-a^2}{(t-a)^2}$.
5. $-\dfrac{c}{t^2}$.
8. $\dfrac{2}{(z+1)^2}$.
10. $-\dfrac{12}{x^2}$.
11. $\dfrac{t^2 - 1}{t^2}$.
12. $\dfrac{1}{(\theta + 1)^2}$.
13. $\dfrac{-2az}{(z^2 + a^2)^2}$.

Page 94

1. *a.* 3. *b.* 3, -3. *d.* $-\frac{4}{3}$. *g.* $-\frac{1}{2}$, $-\frac{8}{25}$, $-\frac{8}{9}$.
2. *a.* (6, 13). *b.* $(\pm 2, \mp \frac{8}{3})$. *c.* $(2, -\frac{7}{3})$, $(-1, \frac{13}{6})$. *d.* $(\pm 2, \pm 4)$.
3. *a.* $y = 3x - 2$, $y = 3x + 2$.
   b. $y = 3x - 12$, $y = -3x + 3$.
   d. $4x + 3y + 24 = 0$.
   g. $x + 2y = 4$, $8x + 9y = 16$, $8x + 25y = 48$.
6. $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$.

Pages 100, 101

2. *a.* $20x^4 - 6x^2$.
   c. $6x - 5$.
   e. $33x^{10} - 63x^8$.
3. *a.* $-\dfrac{6}{x^3} + \dfrac{10}{x^6}$.
   b. $3x^2 + \dfrac{1}{x^2}$.
   c. $40t^4 - \dfrac{21}{t^4} + \dfrac{12}{t^5}$.
   d. $-\dfrac{3}{t^2} + \dfrac{12}{t^3}$.
   e. $\dfrac{20}{x^6} + \dfrac{3}{\sqrt{x}}$.
4. *a.* $1 + \dfrac{2}{x^3}$.
   b. $-\frac{1}{2}x^{-\frac{3}{2}} + \frac{1}{2}x^{-\frac{1}{2}}$.
   c. $4t(t^2 - 1)$.
   d. $3(t-1)^2$.
   g. $-\frac{4}{3}$.
5. *a.* 28.
   b. $\frac{3}{16}$.
   c. 4.494.
   d. $\frac{160}{27}$.
6. *a.* $-\dfrac{3}{x^2} - \dfrac{1}{2}$.
   b. $-\dfrac{18}{x^2}$.

   c. $-\dfrac{5}{x^2}$.
   d. $1 - \dfrac{1}{\sqrt{x}}$.
7. *a.* -1.
   b. $\frac{5}{3}$.
   c. 6.
   d. -2.
   e. $-\frac{32}{27}$.
   f. $\frac{21}{2}$.
   g. -3, $-\frac{4}{3}$, $-\frac{3}{4}$, $-\frac{1}{3}$.
   h. $\pm\frac{1}{2}$.

Pages 107, 108

2. *a.* $14x^6 + 12x^3 + \dfrac{9}{x^4}$.
   b. $\dfrac{3}{(3-5x)^2}$.
   c. $\dfrac{-2x^2 + 2x - 2}{(x^2 - 2x)^2}$.
   d. $5x^{-\frac{1}{6}} + x^{-\frac{1}{2}} + x^{-\frac{2}{3}}$.
   e. $\dfrac{2(x^2-1)}{x^2}$.
   f. $\dfrac{2(t^2-1)}{(1+t+t^2)^2}$.
   g. $3t^2 + 4t - 1$.
   h. $\dfrac{9z^2}{(3-z^3)^2}$.
   i. $\dfrac{4 - 4x + 6x^2}{(1-3x)^2}$.
   j. $\dfrac{1}{\sqrt{x}(\sqrt{x}+1)^2}$.
   k. $\dfrac{-4}{3x^{\frac{2}{3}}(2 + x^{\frac{1}{3}})^2}$.
   l. $\dfrac{-4a^2t}{(a^2+t^2)^2}$.
   m. $\dfrac{2a^2z}{(z^2+a^2)^2}$.
   n. $\dfrac{-1}{(2x-1)^2}$.
   o. $\dfrac{-7}{\sqrt{x}(7+\sqrt{x})^2}$.
   p. $\dfrac{-16a^3x}{(x^2+4a^2)^2}$.
   q. $\dfrac{-32x^3}{(1-x^4)^2}$.
   r. $\dfrac{a^2 - x^2}{(x^2+a^2)^2}$.

3. a. 3.
b. $\frac{3}{8}$.
c. 0.14.
d. $\frac{15}{8}$.
e. -2.
f. -0.1392.

4. -1.

5. $x + 2y = 4$.

6. $(-1, -2)$.

7. $45°$.

8. $0, \pm\frac{1}{4}$.

9. $-2, 3, 6$.

10. $-\frac{1}{27}$.

11. $(1, 3), (-1, -3)$.

12. $-\frac{1}{8}$.

Pages 112, 113

1. a. $\dfrac{4u^7}{\sqrt{x}}$. b. $\dfrac{(1 - 2\sqrt{u})(x+1)}{\sqrt{u}}$. d. $\dfrac{-24}{(2x+9)^2}$.

2. e. $\dfrac{x}{(a^2 - x^2)^{\frac{3}{2}}}$.
f. $-\frac{9}{2}\sqrt{2 - 3x}$.
g. $\dfrac{3x^2}{2\sqrt{1 + x^3}}$.
h. $\dfrac{-(x+2)}{(x^2 + 4x + 10)^{\frac{3}{2}}}$.
i. $\dfrac{3(3x+1)}{2(3x)^{\frac{3}{2}}}$.
j. $\dfrac{2a - 3t}{2t\sqrt{a - t}}$.
k. $-6u(4 - u^2)^2$.
l. $\dfrac{2z}{(1-z)^3}$.

4. a. $\dfrac{-x}{\sqrt{9 - x^2}}$.
b. $\dfrac{-bx}{a\sqrt{a^2 - x^2}}$.
c. $\sqrt{\dfrac{p}{2x}}$.
d. $\dfrac{-\sqrt{a^{\frac{2}{3}} - x^{\frac{2}{3}}}}{x^{\frac{1}{3}}}$.

5. a. $(\pm 1, \pm 2)$.
b. $(0, 0), (1, 1), (2, 0)$.
c. $(0, -1,) (2, -5), (-2, -5)$.

7. $153.43° (153° 26')$.

8. $3x - 4y = 10$.

Page 117

1. a. $8t - \dfrac{2}{\sqrt{t}}$.
b. $\dfrac{4at}{(a - t^2)^2}$.
c. $\dfrac{12x - 5x^2}{2\sqrt{3 - x}}$.
d. $\dfrac{4 + 2t^2}{\sqrt{4 + t^2}}$.
e. $\dfrac{14 - 9x}{2\sqrt{7 - 3x}}$.
f. $\dfrac{1 + t}{(1 + 2t)^{\frac{3}{2}}}$.
g. $\dfrac{a}{(a - x)\sqrt{a^2 - x^2}}$.
h. $(1 - v^2)^{-\frac{3}{2}}$.
i. $\dfrac{9x + 16}{(4 - 3x^2)^{\frac{3}{2}}}$.
j. $\dfrac{3x - 2x^2 - 2x^4}{\sqrt{2 + x^2}\sqrt[3]{(3 - x^3)^2}}$.

2. a. 3.
b. 3.
c. 1.624.
d. 0.
e. ∞.

3. a. $\dfrac{6\sqrt[6]{y^5}}{3\sqrt[3]{y} + 4\sqrt{y}}$.
b. $\dfrac{\sqrt{4 + y^2 - y^4}}{y - 2y^3}$.
c. $\dfrac{\sqrt{4 - y^2}}{-3y}$.

Pages 121, 122

1. *a.* $\dfrac{2x+y}{5y^4-x}$.
   b. $\dfrac{y^2-2xy}{x^2-2xy+3y^2}$.
   c. $-\dfrac{y+1}{3y^2+x+1}$.
   d. $-\dfrac{2x+3y}{3x+2y}$.
   e. $-\dfrac{4\sqrt{x^3y}+y}{4\sqrt{xy^3}+x}$.
   f. $-\dfrac{2\sqrt{xy}+y}{2\sqrt{xy}+x}$.
   g. $-\dfrac{2}{9}\sqrt[3]{\dfrac{9y}{2x}}$.
   h. $\dfrac{ay-x^2}{y^2-ax}$.

8. -1.

9. *c.* $18x^{-4}-24x^{-5}$.
   d. $160t^3+84t^{-5}-60t^{-6}$.
   e. $-120x^{-7}-\frac{3}{2}x^{-\frac{3}{2}}$.
   f. $-\dfrac{6}{x^4}$.
   j. $2x^{-3}+6x^{-4}-12x^{-5}$.
   k. $\frac{3}{4}x^{-\frac{1}{2}}+\frac{15}{4}x^{-\frac{7}{2}}$.
   l. $2+\dfrac{6}{t^4}$.

Pages 125, 126

1. *a.* $(-\frac{3}{2}, -\frac{9}{4})$.
   b. $(-\frac{1}{2}, \frac{17}{4})$.
   c. $(\pm 1, \pm 2)$.
2. $(4, 2)$.
3. $(\pm 2, 1)$, $(2\sqrt{3}, 3)$.
4. $(-\frac{1}{2}, \frac{5}{8})$, $(-\frac{3}{2}, \frac{27}{8})$.
5. *a.* $2x+y=0$, $x-2y=0$.
   b. $2x-y=5$, $x+2y+5=0$.
   c. $x-3y+10=0$, $3x+y=0$.
6. $\sqrt{80}$.
7. $\sqrt{5}$.
8. $\frac{5}{2}\sqrt{17}$.
9. *a.* 30.96° (30° 58′), 108.43° (108° 26′).
   b. 36.87° (36° 52′).
   c. 45°, 135°.
   d. 71.57° (71° 34′), 161.57° (161° 34′).
   e. 71.02° (71° 1′).
   f. 108.43° (108° 26′).
   g. 16.27° (16° 16′), 163.73° (163° 44′).

Pages 131, 132

1. $A=\dfrac{d^2}{2}$.
5. $V=\pi r^2(84-2\pi r)$; 5237, 6783, 6650.
6. $C=\dfrac{1}{2}+\dfrac{500}{n}$.
7. $M=2\pi r^2+\dfrac{116}{r}$; 91.47, 83.13, 85.67.
8. $y=\dfrac{1000m}{\sqrt{1+m^2}}$.
9. $\theta=212-0.002h$.
10. $p=\dfrac{v^2}{225}$.
11. $v=\sqrt{2gs}$.
12. $A=\dfrac{8x}{3}\sqrt{9-x^2}$.
13. When $t=3\frac{1}{8}$ sec.
15. $w=\dfrac{1{,}600{,}000{,}000}{x^2}$; 95.18 lb.
16. $t=0.5547\sqrt{l}$; 0.8125 ft.

Pages 138-140

3. $P=2x+\dfrac{640}{x}$.
9. $A=x\sqrt{25-x^2}$.
11. $C=\dfrac{0.5+1.2n}{n}$.
12. $F=\frac{9}{5}C+32$.
13. $A=x^2+\dfrac{24}{x}$.
16. $V=4x^3-50x^2+150x$.
17. $A=\dfrac{x}{2}\sqrt{256-x^2}$.
18. $P=2x+\sqrt{256-x^2}$.

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6. *a.* Increasing when $t<3\frac{1}{8}$; decreasing when $t>3\frac{1}{8}$.
   b. Always increasing.
   c. Increasing when $0<r<8.93$; decreasing when $r<0$ and $r>8.93$.
   d. Increasing when $x>\sqrt[3]{12}$; decreasing when $x<\sqrt[3]{12}$.
   f. Increasing when $x<1.962$ and $x>6.371$;
   decreasing when $1.962<x<6.371$.

Pages 153–158

1. *a.* Min. $(\frac{5}{2}, -\frac{13}{4})$.
   b. Max. $(\frac{7}{4}, \frac{57}{8})$.
   c. Min. $(2, -\frac{4}{3})$, max. $(-1, \frac{19}{6})$.
   d. Max. $(-3, 2)$, min. $(-5, -2)$.
   e. Min. $(3, -\frac{27}{2})$, max. $(-2, \frac{22}{3})$.
   f. No min. or max.
   g. Min. $(-1, -1)$, $(1, -1)$;
   max. $(0, 0)$.
   h. Min. $(-1, -\frac{38}{15})$, $(2, \frac{16}{15})$;
   max. $(-2, -\frac{16}{15})$, $(1, \frac{38}{15})$.
   i. Min. $(0, -4)$, $(2, -4)$;
   max. $(1, -3)$.
   j. Min. $(0, 0)$, max. $(\pm\sqrt{3}, 9)$.
   k. Min. $(2, 12)$.
   l. Min. $(\pm 1, 2)$.
   m. Min. $(1, 4)$, max. $(-1, -4)$.
   n. Min. $(-1, -3)$, max. $(1, 3)$.
2. *a.* 10, 10.
   b. 10, 10.
   c. 10, 10.
   d. 1, 19.
3. $40\sqrt{2}$ rd. $\times$ $80\sqrt{2}$ rd.
5. $20\sqrt{2}$ in.
6. 3.5 in.
7. 24 ft.
8. 2.94 in.
9. 7.7 ft. $\times$ 7.7 ft. $\times$ 13.5 ft.
10. 4.90 ft. $\times$ 3.27 ft.
11. 550.8 cu. ft.
12. $\frac{a}{3}\sqrt{3} \times \frac{a}{3}\sqrt{6}$.
13. $a \times a\sqrt{3}$.
14. $a\sqrt{2} \times \frac{a}{2}\sqrt{2}$.
15. 100 sq. rd.
16. $a\sqrt{2} \times a\sqrt{2}$.
17. a.
18. 28 in.
19. 28 in.
20. *a.* $\sqrt[3]{\frac{464}{\pi}} = 5.29$ in.
    b. $\sqrt[3]{\frac{232}{\pi}} = 4.20$ in.
21. $10\sqrt[3]{\frac{16}{\pi}} = 17.2$ ft.
22. $\frac{4\pi r^2 h}{27}$.
23. $\frac{a}{3}\sqrt{3}$.
24. $h = \frac{16}{3}\sqrt{3}$ in., $r = \frac{8}{3}\sqrt{6}$ in.
25. $h = \frac{2r}{\sqrt{3}}$.
26. Altitude of cylinder = altitude of cone.
27. 13.6 mi./hr.
28. 75.
29. 16 square units.
30. $a\sqrt{2} \times b\sqrt{2}$.
31. $4a^2$.
32. $(6, 8)$.
33. $(\pm\sqrt{8}, 2)$.
36. $2r$.
37. $AC = 76.14$ ft.
38. 5.18 ft.
41. *a.* $2a$, $2b$.
    b. $a + a^{\frac{1}{3}}b^{\frac{2}{3}}$, $b + a^{\frac{2}{3}}b^{\frac{1}{3}}$.
    c. $a + \sqrt{ab}$, $b + \sqrt{ab}$.
    d. $\frac{a^2+b^2}{a}$, $\frac{a^2+b^2}{b}$.

Pages 162, 163

3. $l = 999.62 + 0.019\,T$.
4. $l = 4 + 0.125\,w$.
6. $p = 759.17 - 0.0833\,h$.
7. 188°.
8. $d = 15 - 0.034\,t$.
9. 0.64 in./min.

Pages 169–171

1. *a*. 5.625.
   b. 3.
   c. 1.89.
2. Approximate change in $y = -1$.
7. 80 ft./sec., 128 ft./sec., 12.96 ft.
9. *a*. $-\frac{26}{27}$ ft. per second per second.
   b. $-\frac{\sqrt{2}}{16}$ ft. per second per second.
11. 2463 cu. in./in., 739 cu. in.
12. 18.8 mi./hr.
13. *a*. 51.5 mi./hr.
    b. 3.45 P.M.
    c. 70.7 mi.
14. 7.9 ft./sec.
16. 0.054 sec./in., 0.0108 sec.
17. $-$ 0.035 lb. per square inch per cubic inch, $-$ 0.35 lb./sq. in.

Pages 174, 175

1. 1.06 ft./sec.
2. When bottom is 16.97 ft. from wall.
3. 0.52 in./sec.
4. *a*. 5 mi./hr.
   b. 2 mi./hr.
5. 3 mi./hr.
6. 6.66 ft./sec.
8. 5.33 sq. in./sec.
9. $\frac{6}{25\pi}$ ft./min.
10. *a*. -156π cu. in./min.
    b. $-$ 118.6 sq. in./min.
11. 10π cu. in./sec.
12. *b*. 1 unit/min.
14. $-$0.1 ft. per second per minute.
15. $-\frac{1}{256}$ units/min.

Page 180

1. *a*. $(1-7x^2)(1-x^2)^2dx$.
   b. $\dfrac{-(2+x)\,dx}{8x^2\sqrt{1+x}}$.
   d. $\dfrac{2\,dy}{(y+1)^2}$.
   e. $\dfrac{2+3t}{2\sqrt{1+t}}dt$.
   f. $\dfrac{(2a-3bx)}{2\sqrt{a-bx}}dx$.
2. *a*. $-\sqrt{\dfrac{y}{x}}\,dx$.
   b. $\dfrac{ay-x^2}{y^2-ax}dx$.
   c. $\dfrac{b^2x}{a^2y}dx$.
   d. $-\dfrac{y}{x}dx$.
   e. $\dfrac{-2xy^3\,dx}{3x^2y^2+4}$.
   f. $-\dfrac{4+y}{x+2y}dx$.
3. 0.031.
4. 2.
6. *a*. $-$ 0.6.
   b. 0.2.
   c. 0.6.
   d. 0.4.
7. 0.01.
8. 1060.

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3. $2\pi rht$.

4. *a.* $a = \dfrac{32.2\,(21{,}000{,}000)^2}{s^2}$.

b. -0.064.

5. Error ≤ 0.0058 in.

6. *a.* 3.26 ft.

b. 0.00153 sec.

c. $-$ 2 min. 12 sec.

7. 1.2, 65.2, 62.8.

8. 0.198.

10. 0.0405.

11. Error ≤ 40.2 cu. in.

12. 1.3 H.P.

13. $\dfrac{1}{18\pi}$ in.

Pages 186, 187

1. Error $\simeq 0.5\%$.

7. 2%.

8. 32.20, 0.022, 0.07%.

9. 493 ft.

Pages 193, 194

2. *a.* $2xy = 1$.

b. $y^2 = 4x$.

e. $y^2 = x^3$.

g. $y^2 = (x-1)^3$.

i. $x^2 - y^2 = 4$.

3. *a.* $v = 3.003$ down, $\alpha = 179.2°$ (179° 12′).

b. $v = 4.472$, $\alpha = 26.57°$ (26° 34′).

c. $v = 6.083$, $\alpha = 80.54°$ (80° 32′).

d. $v = 3.606$, $\alpha = 56.31°$ (56° 19′).

e. $v = 12.65$, $\alpha = 71.57°$ (71° 34′).

f. $v = 4.472$, $\alpha = 153.43°$ (153° 26′).

g. $v = 12.65$, $\alpha = 71.57°$ (71° 34′).

h. $v = 20$, $\alpha = 36.87°$ (36° 52′).

i. $v = 1.458$, $\alpha = 59.04°$ (59° 2′).

j. $v = 1$, $\alpha = 90°$.

4. $v = 16.28$, N. 10.62° E. (N. 10° 37′ E.)

5. $x = 6 + \frac{48}{5}t$, $y = \frac{64}{5}t$.

6. *a.* $v = 1.505$, $\alpha = 131.63°$ (131° 38′).

b. $v = 5.408$, $\alpha = 56.31°$ (56° 19′).

c. $v = 12.17$, $\alpha = 9.46°$ (9° 28′).

d. $v = 2$, $\alpha = 0°$; $v = 2.236$, $\alpha = 153.43°$ (153° 26′).

8. $v_x = 2\sqrt{2}$, $v_y = -2\sqrt{2}$, $\alpha = 135°$.

9. $(-3\sqrt{2}, 3\sqrt{2})$, $(3\sqrt{2}, -3\sqrt{2})$.

10. $\left(2 - \dfrac{\sqrt{3}}{2}, \dfrac{77}{4}\right)$, $\left(2 + \dfrac{\sqrt{3}}{2}, \dfrac{77}{4}\right)$.

11. $v_x = \pm\frac{72}{5}$, $v_y = \mp\frac{96}{5}$.

12. $v_y = 120 - 32.2\,t$.

Pages 200, 201

4. *e.* $x^2 + y^2 \pm 4x - 14y + 49 = 0$.

i. $x^2 + y^2 = 18$.

j. $2x^2 + 2y^2 - 4x - 8y - 15 = 0$.

6. *a.* Length $5\sqrt{2}$.

b. Length $2\sqrt{5}$.

8. $2x^2 + 2y^2 = k - 2c^2$.

9. $(x + 3c)^2 + y^2 = 4k^2$ or $(x - 3c)^2 + y^2 = 4k^2$.

10. $x^2 + y^2 - 16x + 48 = 0$.

11. $x^2 + y^2 - 5x + 5y + 8 = 0$.

12. $C = \left(\dfrac{c(1+k^2)}{1-k^2}, 0\right)$, $r = \dfrac{2kc}{k^2 - 1}$.

13. *a.* $4x - 3y = 8$, $3x + 4y = 6$.

b. $y = -2$, $y = -4$; $x = 4$.

c. $5x \pm 12y = -52$, $12x \pm 5y = 78$.

14. *a.* 45°. *e.* 45°.

b. 90°. *f.* 66.25° (66° 15′).

c. 45°. *g.* 71.57° (71° 34′).

d. 45°.

15. *a.* 63.43° (63° 26′).

b. 90°.

16. 60°.

17. 60°.

18. 63.43° (63° 26′).

Pages 207, 208

2. *e.* $x^2+y^2-20x-8y+16=0$ or $x^2+y^2-52x+24y+144=0$.
   f. $x^2+y^2-4x-4y=0$.
   g. $x^2+y^2-26x+26y+169=0$.
   h. $x^2+y^2+4x-8y=52$.
3. *a.* $(2, -3)$, $(3, 2)$.
   b. $(0, 0)$, $(8, 4)$.
4. 38.
5. Length $a\sqrt{2}$, width $\frac{a}{\sqrt{2}}$.
6. *a.*
8. 5.81×2.07.
11. $45°$.
12. *a.* $(x-2)^2+(y+2)^2=25$ or $(x-2)^2+(y+2)^2=225$.
    b. $x^2+y^2+14x+4y+17=0$ or $x^2+y^2+38x+4y+329=0$.

Pages 213, 214

3. *a.* $x^2+16y=0$.
   e. $y^2=-18x$.
   f. $y^2=9x$ or $3x^2=8y$.
4. $(\frac{1}{4}, 1)$.
5. $x^2+y^2-10x=0$.
6. $0°$, $8.13°(8°8')$.
9. *a.* $x-4y+6=0$, $4x+y=27$.
   b. $x-y=4$, $x+y=12$.
10. *a.* $(\frac{2}{3}, 1)$.
    b. $(4, 1)$.
    c. $(-\frac{9}{5}, -3)$.
    d. $(\frac{1}{2}, \frac{3}{2})$.
12. *a.* $12.53°(12°32')$, $63.43°(63°26')$.
    b. $76.1°(76°6')$.
    c. $108.43°(108°26')$.
13. $\tan\theta=\frac{3}{4}$.
14. 13.86×4.

Pages 220, 221

4. *a.* $x^2-4x+16y-44=0$.
   b. $x^2-8x-12y+28=0$.
   c. $y^2-6y+16x-23=0$.
   d. $y^2-4y-20x+24=0$.
5. Vertex $\left(\frac{a}{4}, \frac{a}{4}\right)$.
6. $(2, 4)$.
7. $(5, -7)$.
9. $18.92°(18°56')$.
10. $4x+4y=25$.
11. $71.57°(71°34')$, $30.96°(30°58')$.
12. $\left(4\pm\frac{1}{2}\sqrt{62}, \frac{1}{2}\right)$.
13. $h=r$.
15. 60, 47.8, 37.8, 30, 24.4, 21.1, 20, 21.1, etc.

Pages 227, 228

7. $x^2+4y^2=64$.
8. $b^2x^2+a^2y^2=a^2b^2$.
10. $x-2y=7$, $2x+y=4$.
11. $x+2y=6$, $x+14y=30$.
12. $(0, 8)$.
13. *a.* $90°$.
    b. $42.08°(42°5')$.
    c. $18.43°(18°26')$, $21.37°(21°22')$.
    d. $71.57°(71°34')$.
    e. $100.46°(100°27')$.
14. $(3, \frac{16}{3})$, $(-3, -\frac{16}{3})$.
15. Speed 2.062, $\alpha=165.96°(165°58')$.
17. $2ab$.
18. $2y+b=0$.

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4. $25x^2 - 144y^2 + 3600 = 0$.

6. *a.* $5x - 4y = 9, 4x + 5y = 40$.
   b. $16x + 5y = 39$,
   $5x - 16y = 100$.

7. $2x - y = \pm 3$.

8. $3x - 5y \pm 16 = 0$.

10. $22.62°\,(22° 37')$.

13. Decreasing $\frac{9}{4}$ units/sec.

14. 2.54×7.44.

15. $(5, 3)$.

Pages 237, 238

3. $4x^2 - y^2 = 3$.

4. $-\dfrac{a + y}{b + x}$.

6. $(8, 6), (-8, -6)$.

7. *a.* $x + y = 4, x - y + 12 = 0$.
   b. $3x - 2y = 24, 2x + 3y + 10 = 0$.
   c. $3x - 2y + 25 = 0, 2x + 3y = 18$.

8. 5.19.

11. *a.* $49.4°\,(49° 24')$.
    b. $72.26°\,(72° 15')$.
    c. $30.96°\,(30° 58')$.
    d. $16.26°\,(16° 16')$.

14. $k = \pm 8$.

Pages 249, 250

2. Max. pt. $\left(\sqrt{5}, \dfrac{3 + 10\sqrt{5}}{3}\right)$, min. pt. $\left(-\sqrt{5}, \dfrac{3 - 10\sqrt{5}}{3}\right)$, infl. pt. $(0, 1)$, slope 5.

4. Max. pt. $(-1, \frac{1}{2})$, min. pt. $(0, 0)$, infl. pt. $(-\frac{1}{2}, \frac{1}{4})$, slope $-\frac{3}{4}$.

6. Max. pt. $\left(1 - \dfrac{\sqrt{3}}{3}, \dfrac{2\sqrt{3}}{9}\right)$, min. pt. $\left(1 + \dfrac{\sqrt{3}}{3}, -\dfrac{2\sqrt{3}}{9}\right)$, infl. pt. $(1, 0)$, slope -1.

8. Max. pt. $(0, -\frac{3}{5})$, min. pts. $(\pm 2\sqrt{3}, -\frac{39}{5})$, infl. pts. $(\pm 2, -\frac{23}{5})$, slopes $\mp \frac{16}{5}$.

10. Max. pt. $(0, 4)$, min. pts. $(\pm \frac{1}{2}\sqrt{10}, -\frac{9}{4})$, infl. pts. $(\pm \frac{1}{6}\sqrt{30}, \frac{19}{36})$, slopes $\mp \frac{10}{9}\sqrt{30}$.

12. Max. pts. $(1, \frac{38}{15}), (-2, -\frac{16}{15})$; min. pts. $(-1, -\frac{38}{15}), (2, \frac{16}{15})$; infl. pts. $(0, 0)$, $\left(\pm \dfrac{1}{2}\sqrt{10}, \pm \dfrac{13\sqrt{10}}{24}\right)$; slopes $4, -\frac{9}{4}$.

14. Min. pt. $(1, 3)$, infl. pt. $(-\sqrt[3]{2}, 0)$, slope $-3\sqrt[3]{2}$.

16. Min. pt. $(\sqrt[5]{\frac{2}{3}}, \frac{5}{3}\sqrt[5]{\frac{9}{4}})$, infl. pt. $(-1, 0)$, slope 5.

18. $(0, 0), (\pm\sqrt{3}, \pm\frac{1}{4}\sqrt{3})$.

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2. Min. pt. $(0, 2)$, max. pts. $(\pm\sqrt{12}, 14)$, infl. pts. $(\pm 2, \frac{26}{3})$.

4. Min. pt. $(0, 0)$, infl. pts. $\left(\pm \dfrac{\sqrt{3}}{3}, \dfrac{1}{4}\right)$, asympt. $y = 1$.

6. Max. pts. $(\pm\sqrt{2}, 4)$, min. pt. $(0, 0)$, infl. pts. $\left(\pm \dfrac{\sqrt{6}}{3}, \dfrac{20}{9}\right)$.

8. Min. pt. (2, 3), asympt. $x = 0$.

9. Max. pt. $\left(-2\sqrt[5]{\frac{2}{3}}, -\frac{5}{3}\sqrt[5]{\frac{9}{4}}\right)$, infl. pt. (2, 0), asympt. $x = 0$.

12. Max. pt. $(0, 2a)$, infl. pts. $\left(\pm\frac{2a}{\sqrt{3}}, \frac{3a}{2}\right)$, asympt. $y = 0$.

14. Max. pt. $\left(-\frac{3}{5}, \frac{162}{3125}\right)$; min. pt. $\left(\frac{3}{5}, -\frac{162}{3125}\right)$; infl. pts. (0, 0), $\left(\pm\frac{3\sqrt{2}}{10}, \mp\frac{567\sqrt{2}}{25000}\right)$.

16. Max. pt. (0, 0), min. pt. $\left(\sqrt[3]{\frac{2}{5}}, -\frac{3}{25}\sqrt[3]{20}\right)$, infl. pt. $\left(\sqrt[3]{\frac{1}{10}}, -\frac{9}{100}\sqrt[3]{10}\right)$.

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2. Max. pt. $\left(\frac{\sqrt{3}}{3}, \frac{\sqrt[4]{12}}{3}\right)$, min. pt. $\left(\frac{\sqrt{3}}{3}, -\frac{\sqrt[4]{12}}{3}\right)$.

4. No max. or min. pts.

6. Max. pts. $(\pm 2\sqrt{2}, 8)$, min. pts. $(\pm 2\sqrt{2}, -8)$.

8. Max. pt. $\left(-\frac{2\sqrt{3}}{3}, \frac{4\sqrt{3}}{3}\right)$, min. pt. $\left(\frac{2\sqrt{3}}{3}, -\frac{4\sqrt{3}}{3}\right)$.

10. No max. or min. pts., asympt. $x = 2a$.

12. Max. pt. $x = \frac{16}{25}$, infl. pt. $x = \frac{64}{225}$.

14. Asympt. $x = 1$, $x = 3$.

16. Max. pt. (0, 4), min. pt. (0, − 4).

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1. *a.* $R = 4\sqrt{2}$.
   b. $R = \frac{1}{6}$.
   c. $R = \frac{13\sqrt{13}}{6}$.
   d. $R = \frac{5\sqrt{5}}{2}$.
   e. $R = \frac{10\sqrt{5}}{3}$.
   f. $R = \frac{13\sqrt{13}}{6}$.
   g. $R = 4\sqrt{2}$.
   h. $R = \frac{1}{8}$.
   i. $R = \frac{1}{2}$, $R = \frac{\sqrt{2}}{2}$.
   j. $R = 2p\sqrt{2}$.

2. *a.* $\frac{(b^4x^2 + a^4y^2)^{\frac{3}{2}}}{a^4b^4}$.
   b. $\frac{(b^4x^2 + a^4y^2)^{\frac{3}{2}}}{a^4b^4}$.
   c. $\frac{(1 + 9a^2x^4)^{\frac{3}{2}}}{6ax}$.
   d. $\frac{(4 + 9ax)^{\frac{3}{2}}}{6}\sqrt{\frac{x}{a}}$.

6. $\frac{a\sqrt{2}}{2}$.

7. *a.*

8. *a.* $\left(\sqrt[4]{\frac{1}{45}}, \sqrt[4]{(\frac{1}{45})^3}\right)$.
   b. (2, 4).
   c. $(2\sqrt{6}, 2\sqrt{6})$.
   d. No point.

9. *a.* $\frac{17\sqrt{17}}{8}$.
   b. $\frac{13\sqrt{6}}{6}$.
   c. $\frac{73\sqrt{73}}{48}$.

Pages 273, 274

16. $\frac{3}{2}x^{\frac{4}{3}} - \frac{3}{7}x^{\frac{7}{3}} + C.$

17. $-\frac{4}{3x^3} + \frac{1}{x} + C.$

18. $ax - \frac{4}{3}x\sqrt{ax} + \frac{x^2}{2} + C.$

19. $a^2x - \frac{9}{5}ax\sqrt[3]{ax^2} + \frac{9}{7}x^2\sqrt[3]{a^2x} - \frac{x^3}{3} + C.$

20. $6\sqrt{x} - 3\sqrt[3]{x^2} + \frac{1}{x} + C.$

21. $-\frac{2}{3}(1-x)^{\frac{3}{2}} + C.$

22. $-\frac{1}{2}\left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3 + C.$

23. $\frac{2}{9b}(a + bx^2)^{\frac{9}{4}} + C.$

24. $\frac{8}{3}\left(\frac{y}{8} - 3\right)^3 + C.$

25. $\frac{3}{8}(2ax + x^2)^{\frac{4}{3}} + C.$

26. $\frac{x^5}{5} - 2x^3 + 12x + \frac{8}{x} + C.$

27. $-\frac{9}{4}(1 - 2s^2)^{\frac{2}{3}} + C.$

28. $\frac{2x}{3}\sqrt{5x} + C.$

29. $\frac{2}{3}\sqrt{3x} + C.$

30. $\frac{2}{3}(x-2)^{\frac{3}{2}} + C.$

31. $-2\sqrt{5-x} + C.$

32. $\frac{1}{9}(3x-2)^3 + C.$

33. $-\frac{(5-x)^4}{4} + C.$

34. $-\frac{1}{9}(4-3x)^3 + C.$

35. $\frac{1}{2}x^2 + \frac{2}{5}x^{\frac{5}{2}} + C.$

36. $\frac{x^3}{6} - \frac{2}{x} + C.$

37. $\frac{14}{3}x\sqrt{x} + 14\sqrt{x} + C.$

38. $-\frac{1}{x} - \frac{2}{\sqrt{x}} + C.$

39. $-\frac{5}{2(x^2-3)} + C.$

40. $\frac{2}{5}x^2\sqrt{3x} + C.$

41. $\frac{3}{7}x^2\sqrt[3]{2x} + C.$

42. $2x\sqrt{\frac{x}{5}} + C.$

43. $-\frac{1}{3}(a^2-x^2)^{\frac{3}{2}} + C.$

44. $\frac{1}{6}(2s^2-3)^{\frac{3}{2}} + C.$

45. $-7\sqrt{6-t^2} + C.$

46. $\frac{1}{3}(y^4 + a^4)^{\frac{3}{2}} + C.$

47. $\frac{3}{16(9-4v^2)^2} + C.$

48. $\frac{2}{9}(9 + z^3)^{\frac{3}{2}} + C.$

Pages 275, 276

1. $\frac{2}{5}(a-x)^{\frac{5}{2}} - \frac{2a}{3}(a-x)^{\frac{3}{2}} + C.$

2. $\frac{2}{3}(x-2)\sqrt{1+x} + C.$

3. $\sqrt{1+x^2} + C.$

4. $25x - \frac{10}{3}x^3 + \frac{1}{5}x^5 + C.$

5. $\frac{2}{105}(15x^2 - 12x + 8)(1+x)^{\frac{3}{2}} + C.$

6. $\frac{1}{3}(1+x^2)^{\frac{3}{2}} + C.$

7. $x + x^3 + \frac{3}{5}x^5 + \frac{1}{7}x^7 + C.$

8. $-\frac{2}{15}(8 + 4x + 3x^2)\sqrt{1-x} + C.$

9. $-\frac{2}{3}\sqrt{1-x^3} + C.$

10. $\frac{2(bt - 2a)\sqrt{a+bt}}{3b^2} + C.$

11. $\frac{2(a+by)^{\frac{5}{2}}}{5b^2} - \frac{2a(a+by)^{\frac{3}{2}}}{3b^2} + C.$

12. $\frac{1}{3b}(a + by^2)^{\frac{3}{2}} + C.$

13. $\frac{2}{5}x^{\frac{5}{2}} - \frac{8}{3}x^{\frac{3}{2}} + 8x^{\frac{1}{2}} + C.$

14. $\frac{2x\sqrt{2px}}{3} + C.$

15. $\frac{2}{3b}(a + bx)^{\frac{3}{2}} + C.$

16. $-\frac{16}{3}(2-t)^{\frac{3}{2}} + \frac{24}{5}(2-t)^{\frac{5}{2}} - \frac{12}{7}(2-t)^{\frac{7}{2}} + \frac{2}{9}(2-t)^{\frac{9}{2}} + C.$

17. $\frac{(10x^2 - 6x + 3)(4x+3)^{\frac{3}{2}}}{140} + C.$

18. $-\frac{3}{4}(8-x)^{\frac{4}{3}} + C.$

Pages 278, 279

2. $y = \frac{2}{3}(x^{\frac{3}{2}} - 5)$.
3. $y = \frac{1}{3}(x^2 + 16)^{\frac{3}{2}} - \frac{110}{3}$.
5. $y = \frac{3}{4}x\sqrt[3]{x} + \frac{3}{2}\sqrt[3]{x^2} - 18$.
6. $s = \frac{2t}{3}\sqrt{7t}$.
8. $s = \frac{1}{3}t^3 - \frac{4}{3}t^{\frac{3}{2}} + 7$.
9. $y = \frac{2}{5}(x - 6)(9 + x)^{\frac{3}{2}} + \frac{324}{5}$.
10. $y = \frac{2}{3}(x - 10)\sqrt{5 + x} + 13$.
12. $y = -\frac{1}{3}(25 - x^2)^{\frac{3}{2}} + \frac{125}{3}$.
13. $s = 4 - 2\sqrt{5 - x}$.
15. $y = \frac{2(x-2)\sqrt{1+x}}{3} + 2$.
16. $z = \frac{3}{2}x^{\frac{4}{3}} - \frac{3}{7}x^{\frac{7}{3}} + \frac{209}{7}$.
17. 4.
18. -0.8856.
19. $\frac{8p^2}{3}$.
20. $-\frac{66}{35}$.
21. 13.
22. 0.
23. $\frac{49}{6}$.
24. $\frac{298}{15}$.

Pages 281, 282

1. *b.* $2\sqrt{y} = x$.
   c. $3y = 16 - 2(4 - x)^{\frac{3}{2}}$.
   e. $3\sqrt{y} = x^{\frac{3}{2}} + 2$.
   f. $xy = -1$.
   g. $4x^2 + 9y^2 = 324$.
   h. $4y^{\frac{3}{2}} = 3(x^2 - 1)$.
   j. $y = \frac{b}{a}\sqrt{a^2 - x^2}$.
   k. $y = \sqrt{x^2 + 4}$.
   m. $\sqrt{x} + \sqrt{y} = \sqrt{a}$.
   n. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
2. *b.* $y^2 - 4x^2 = C$.
   c. $4x^2 + y^2 = C$.
   e. $y^2 = x + C$.
   f. $3y = x^3 - 3x + C$.
   h. $(x - 4)^2 + (y + 2)^2 = C$.
   i. $\frac{1}{x} - \frac{1}{y} = C$.
5. $2y = 2x - 3x^2$.
6. $y = \frac{4}{x^2}$.
7. $3y = 3x^3 + 2x - 6$.
8. $y = 4x^{\frac{5}{2}} - 79x + 188$.
9. $y = \frac{3}{x} + 2x - 4$.

Pages 285–287

1. 87.64 ft./sec.
2. 89.89 ft./sec.
3. *a.* 12,500 ft.
   b. 1924 ft./min.
   c. 6944 ft.
4. 904 ft.
5. $7\frac{1}{16}$ sec.
6. 126.7 ft.
8. *a.* $v = t^2 - \frac{t^3}{3} + 2$.
   b. $v = \frac{41}{6} - \frac{1}{t} - \frac{t^2}{2}$.
   c. $v = 2 + 2\sqrt{3} - \frac{6}{\sqrt{t}}$.
   d. $v = 4t - \frac{t^3}{3} - 1$.
9. *a.* $s = \frac{2}{3}(t - 1)^{\frac{3}{2}} + 2$.
   b. $s = \frac{t^3}{3} - \frac{1}{t} + \frac{8}{3}$.
11. 4.5 ft.
12. $4\sqrt{10}$ ft./sec.

Pages 292, 293

1. 2.
2. $\frac{16}{3}$.
4. $\frac{38}{3}$.
5. 13.73.
7. $\frac{50}{3}$.
8. $\frac{2}{15}$.
10. $-42\frac{2}{3}$.
11. $-\frac{27}{4}a^{\frac{4}{3}}$.
13. $\frac{1}{3}a^3$.
14. 8.
16. 399.9.
17. 6.95.
19. $\dfrac{a^2}{6}$.
20. $\frac{52}{9}$.
22. $\frac{15}{32}$.
23. $\frac{2}{3}a\sqrt{a}$.
25. 4.
26. $\dfrac{1}{4a^2}$.
28. $-\frac{2}{3}$.
29. $\frac{26}{3}$.

Pages 298, 299

1. *a.* $\frac{16}{3}$. *c.* $\frac{81}{4}$. *e.* $\frac{14}{3}$. *g.* 7. *i.* 14.48. *k.* $\frac{27}{4}$.
2. *a.* 30. *c.* $\frac{28}{3}$.
3. *a.* $\frac{32}{3}$. *c.* $\frac{1}{2}$. *e.* $-\dfrac{128\sqrt{2}}{15}$.
4. $\frac{8}{3}$.
5. $\frac{64}{3}$.
6. 1.
7. $\dfrac{a^2}{6}$.
8. $\frac{2}{3}p^2$.
9. $\frac{2}{3}p^2$.
11. $\frac{32}{3}$.
12. $\dfrac{16\sqrt{2}}{3} = 7.54$.
13. 226.8.

Pages 300, 301

1. *a.* $\dfrac{4a^2\sqrt{a}}{15}$. *b.* $\frac{76}{15}$. *c.* $\frac{26}{3}$. *d.* 87.77. *e.* $-\frac{300}{7}$. *f.* $\dfrac{2a}{3}(2\sqrt{a} - \sqrt{2a})$.
2. $\frac{128}{15}$.
3. $2\sqrt{3}$.
4. $\frac{648}{5}$.
5. $\frac{32}{15}$.
6. $\dfrac{32\sqrt{2}}{15}$.
7. $\frac{2048}{105}$.

Pages 310, 311

2. Exact value 12.87 +.
3. Exact value 0.8813 +.
4. *a.* 16.48. *b.* 1.791. *c.* 5.502. *d.* 45.25. *e.* 17.08. *f.* 65.28.

Page 313

1. *a.* 9.84. *b.* 36.39. *c.* 6.89. *d.* 18.10.
2. *a.* 9π. *c.* $42\frac{2}{3}$. *d.* 72. *e.* $\frac{4}{5}$. *g.* 4.184.
3. 9.2 acres.

Pages 318, 319

1. *a.* 8. *c.* $\frac{9}{2}$. *e.* 9. *g.* $\frac{32}{3}$. *i.* $\frac{4096}{75}$.
2. $\frac{4}{3}p^2$.
3. $\dfrac{ab}{3}$.
4. 36.
5. $\dfrac{a^2}{6}$.
6. 36.
7. 36.
8. $\frac{27}{10}$.
9. $\frac{7}{6}$.
10. 16.64.
11. 44.02.
12. $\frac{128}{5}$.
13. $\frac{256}{15}$.
14. $\frac{3}{4}$.
15. 108.

Pages 322, 323

1. $\frac{335}{27}$.
2. $\frac{14}{3}$.
3. $1.44\,a$.
4. 9.07.
5. $6\,a$.
6. 9.07.
7. $\frac{14}{3}$.
8. $x\sqrt{1+m^2}$.
9. 4.65.
10. 9.30.
11. 8.61.
12. 4.56.
13. 9.38.

Pages 328–330

1. *a.* $\dfrac{32\,\pi}{5}$. *b.* $\dfrac{8\,\pi}{3}$. *c.* $\dfrac{224\,\pi}{15}$. *d.* $8\,\pi$. *e.* $8\,\pi$. *f.* $\dfrac{256\,\pi}{15}$. *g.* $\dfrac{40\,\pi}{3}$. *h.* $\dfrac{128\,\pi}{5}$.

2. *a.* $\dfrac{32\,\pi}{7}$. *b.* $\dfrac{8\,\pi}{5}$. *c.* $\dfrac{80\,\pi}{7}$. *d.* $\dfrac{32\,\pi}{5}$. *e.* $\dfrac{48\,\pi}{5}$. *f.* $\dfrac{144\,\pi}{7}$. *g.* $\dfrac{72\,\pi}{5}$. *h.* $\dfrac{192\,\pi}{7}$.

8. $\frac{4}{3}\,\pi a^2 b$.
10. $\frac{4}{3}\,\pi a^3$.
11. $\dfrac{32\,\pi a^3}{105}$.
12. $\dfrac{\pi a^3}{15}$.
14. $\dfrac{416\,\pi}{3}$ cu.in.
16. $18\,\pi$.
17. $\dfrac{128\,\pi}{15}$.
18. $4\,\pi$.
19. 57.44.
20. $\dfrac{1250\,\pi}{3}$.

Pages 334–336

a. 2309 cu.in.
b. 1333 cu.in.
c. 2667 cu.in.
d. 1571 cu.in.
e. 2667 cu.in.
a. 1333 cu.in.
b. 577.3 cu.in.
c. 392.7 cu.in.

3. *a.* 1024 cu.in. *b.* 443.4 cu.in. *c.* 426.7 cu.in.
4. *a.* 228.7 cu.in. *b.* 359.2 cu.in.
5. $\dfrac{16\,r^3}{3}$.
6. 1066.
7. $\frac{250}{3}$ cu. in.
8. *a.* $\dfrac{625\,\pi}{3}$. *b.* $\dfrac{128\,\pi}{5}$. *c.* $\dfrac{128\,\pi}{105}$. *d.* $\dfrac{64\,\pi}{3}$.
9. *a.* $\dfrac{108\,\pi}{5}$. *b.* $\dfrac{256\,\pi}{15}$. *c.* $\dfrac{12\,\pi p^3}{5}$. *d.* $\dfrac{45\,\pi}{2}$.

Pages 338–340

$\dfrac{13\,\pi}{3}$.
$\dfrac{52\,\pi p^2}{3}$.
203.0.

4. 36.18.
5. $\frac{12}{5}\,\pi a^2$.
6. 53.15.
7. 77.34.
8. 217.1.
10. 410.3
11. 141.5.
13. 131.2.
14. $\dfrac{208\,\pi}{9}$.
15. $3\,\pi$.
16. $\dfrac{56\,\pi\sqrt{3}}{5}$.

Page 345

$(\frac{3}{5}h, 0)$.
$(\frac{8}{5}, \frac{16}{7})$.
6. $(\frac{20}{7}, \frac{5}{2})$.
8. $(\frac{16}{15}, \frac{64}{21})$.
10. $(\frac{12}{5}, \frac{3}{2})$.
12. $(\frac{8}{5}, 1)$.
14. $(1, \frac{17}{5})$.
15. $(2, 1)$.
16. $\left(\dfrac{a}{3}, \dfrac{b}{3}\right)$.
17. $\left(\dfrac{a}{5}, \dfrac{a}{5}\right)$.
18. $(\frac{16}{7}, 0)$.
19. $\left(\dfrac{4\,a}{3\,\pi}, \dfrac{4\,b}{3\,\pi}\right)$.

Page 346

1. $\frac{3}{8}r$.
2. $\frac{2r}{\pi}$.
3. $\frac{r}{2}$.

Pages 350, 351

1. 2560 lb.
2. 2160 lb.
3. 38,229 lb.
4. 1302 lb.
5. 3771 lb.
6. 7800 lb.
7. 533 lb.
8. 1667 lb.
9. 3375 lb.
10. 20,333 lb.
11. *a.* 41,250 lb.
    b. 11 ft.
12. 3897 T.
14. 3682 T.

Pages 354, 355

1. $800{,}000\,\pi$ ft.-lb.
2. $84{,}375\,\pi$ ft.-lb.
3. $432{,}000\,\pi$ ft.-lb.
4. $\frac{2{,}500{,}000\,\pi}{3}$ ft.-lb.
5. $9375\,\pi$ ft.-lb.
6. 78,000 ft.-lb.
7. 327,600 ft.-lb.
8. 13,333 ft.-lb.
9. 8000 ft.-lb.
10. 29,333 ft.-lb.

Page 358

3. *a.* 23.60.
4. *a.* 23.57.

Page 365

1. *a.* 5.
   b. $\frac{3}{2}$.
2. *a.* $x = \log_3 y$.
   b. $x = -\log_2 y$.
   c. $x = -\log y$.
   g. Impossible.
3. *a.* $x = 3^y$.
   b. $6x = e^{2y}$.
7. *a.* $10^{0.008686\,t}$.
8. *a.* $e^{-2.303\,x}$.
9. *a.* 6.227.
   c. 34739.
10. *a.* $x = \log\left(y \pm \sqrt{y^2 - 1}\right)$.
    b. $x = \log\left(y \pm \sqrt{y^2 + 1}\right)$.

Pages 369, 370

1. $y' = \dfrac{a}{ax + b}$.
2. $y' = \dfrac{16\,x}{x^4 - 16}$.
4. $y' = 0.4343\,\dfrac{2 - 4\,x^3}{2\,x - x^4}$.
6. $f'(x) = \dfrac{3}{x}$.
7. $f'(x) = \dfrac{3}{x}\log^2 x$.
9. $f'(x) = \dfrac{1}{\sqrt{1 + x^2}}$.
11. $\dfrac{ds}{dt} = \dfrac{1}{2}(1 + \log t)$.
13. $f'(y) = \dfrac{1}{1 - y^2}$.
17. $f'(x) = \dfrac{x + 3}{2\,(x^2 - 1)}$.
18. $\dfrac{dy}{dx} = \dfrac{1 - \log x}{x^2}$.
19. $\dfrac{dy}{dx} = \dfrac{4 + 5\,x^2}{x\,(1 + x^2)}$.
21. *a.* $\frac{1}{12}$.
    b. 3.
    c. 4.797.
    d. $-$ 0.0483.
    e. 0.2545.
    f. $-$ 0.0362.
22. *a.* 0.3679.
    b. 1.2067.
    c. 0.5067.
    d. 0.5067.
23. *a.* 1, $\frac{1}{2}$.
    b. $\pm\,4\sqrt{2}$, 0.
    c. $\pm\,2$, $\frac{2}{3}$.
    d. 0.43429.
    e. -1, $-\frac{1}{4}$.
    f. $\pm\sqrt{3}$, 0.
    g. $-\frac{1}{2}$, $-\frac{1}{6}$.
    h. $\pm\,0.8686\sqrt{15}$, 0.
24. *a.* (4, 2.7726).
    b. (2, 1.3863).

26. $R = 2.828$.
27. *a.* Min. pt. $(0, 0)$, infl. pts. $(\pm 1, \log 2)$.
b. Min. pt. (e, e), infl. pt. $(e^2, \frac{1}{2} e^2)$.
c. Min. pt. $\left(\frac{1}{e}, -\frac{1}{e}\right)$.
d. Max. pt. $(4, \log 16)$.
28. 2.323.

Page 372

1. $y' = 2(x+2)(x+3)^2(3x^2 + 11x + 9)$.
3. $y' = \dfrac{1 + 3x^2 - 2x^4}{(1-x^2)^{\frac{3}{2}}}$.
5. $y = 6\sqrt{5}$, $y' = \frac{67}{20}\sqrt{5}$.
6. $y = 2\sqrt{6}$, $y' = \frac{23}{12}\sqrt{6}$.
7. $y = 3$, $y' = -\frac{1}{4}$.
8. $y = 49\sqrt{6}$, $y' = \frac{921}{24}\sqrt{6}$.

Pages 374, 375

1. $y' = ae^{ax}$.
3. $y' = -4a^{-4x}\log a$.
5. $y' = \frac{1}{x} a^{\log x} \log a$.
7. $y' = 2x \cdot 10^{x^2} \log 10$.
9. $y' = \dfrac{2e^x}{(e^x + 1)^2}$.
11. $y' = \dfrac{1}{1 + e^x}$.
13. $y' = (x+1)e^x$.
15. $y' = -4xe^{-x^2}$.
17. $u' = \dfrac{1 - w \log w}{we^w}$.
18. $y' = x^x(1 + \log x)$.
22. $2\sqrt{ab}$.
24. Max. pt. $(0, 1)$, infl. pts. $\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right)$.
26. $(0, a)$.
27. $(0, 0)$.
28. Min. pt. $\left(-1, -\frac{1}{e}\right)$, infl. pt. $\left(-2, -\frac{2}{e^2}\right)$.
29. *a.* $R = 2\sqrt{2}$.
b. $R = \frac{1}{2}$.
c. $R = a$.

Page 379

1. $(B - kA - kBt)e^{-kt}$.
2. 1127, 67.62 per hr.
3. $C = 30$, $k = 0.0000385$, $-0.0000385\,p$, 3720 ft.

Pages 381–383

1. $\dfrac{e^{ax}}{a} + C$.
3. $\frac{1}{2} e^{2s} + C$.
5. $\dfrac{a^{2x}}{2 \log a} + C$.
8. $\frac{1}{2}x^2 - x + \log(x+1) + C$.
9. $\log\sqrt{x^2 - 2x - 5} + C$.
11. $2y^{-4} - 6y^{-2} + \frac{1}{2}y^2 - \log y^6 + C$.
13. $\frac{1}{4}\log^4 x + C$.
14. $\log(e^x + 1)^2 + C$.
16. $\frac{1}{2}(e^{2y} - e^{-2y}) + 2y + C$.
19. 3.195.

20. 0.5596.
21. 1.
22. 25.94.
23. 1.1513.
24. $\log\sqrt{2}$.
25. $\frac{8}{3} - \log 3$.
26. 0.3167.
27. 8.318.
28. 8.318.
30. 0.3181.
31. $e^x - 1$.
32. $a^2\left(e - \frac{1}{e}\right)$.
33. 1.070.
34. 5.751.
35. $\frac{a}{2}\left(e - \frac{1}{e}\right)$.
36. 4.443.
37. $2.594\,\pi$.
38. $0.2115\,\pi a^3$.
39. $\frac{\pi}{2}(e^2 - 1)$.
40. $\pi\left(\frac{15}{2} - 4\log 4\right)$.
41. $\frac{\pi a^3}{4}(e^2 + 4 - e^{-2})$.
42. $\frac{\pi}{2}(1 - e^{-20})$.

Page 385

1. $y = \log(1 + x^2)$.
2. $xy = C$.
3. $y = ce^{\frac{x^2}{2}}$.
4. $y = e^{\frac{x^2}{2} - 2}$.
5. $y = ce^x$.
6. $I = Ce^{-0.02x}$.
7. 34.7 mm.
8. $v = 1000\,e^{-0.25t}$, 82.08 rev./sec.
9. $p = 15\,e^{-0.00004h}$.
10. $L = 60\,e^{0.00001\,T}$.

Pages 395–397

1. *a.* $y' = 2\cos 2x$.
c. $\frac{dr}{dt} = -2a\sin 2t$.
e. $f'(y) = 2\cos 2y\cos 3y - 3\sin 2y\sin 3y$.
f. $f'(\theta) = \theta\cos\theta$.
g. $f'(x) = \sin 2x$.
i. $f'(x) = -\tan x$.
j. $f'(x) = \frac{1}{2}\cot x$.
l. $y' = \frac{\cos\sqrt{x}}{2\sqrt{x}}$.
m. $\frac{dr}{d\theta} = \frac{-\sin\theta}{2\sqrt{\cos\theta}}$.
o. $\frac{ds}{dt} = -e^{-t}(\sin t + \cos t)$.
q. $y' = \frac{\cos x + x\sin x}{\cos^2 x}$.

2. *a.* 1.081.
b. -3.637.
c. 1.
d. -0.2580.
e. 1.382.
f. -1.

3. *a.* $y'' = -\frac{1}{9}\sin\frac{x}{3}$.
b. $r'' = -18\cos 3\theta$.
c. $y'' = -4(\sin 2x + x\cos 2x)$.
d. $s'' = -e^{-t}(4\cos 2t + 3\sin 2t)$.

4. *a.* Max. pts. $\left(\frac{\pi}{2} \pm 2n\pi,\ 1\right)$, min. pts. $\left(-\frac{\pi}{2} \pm 2n\pi,\ -1\right)$.

5. *a.* $(\pm n\pi,\ 0)$.
6. *a.* $y = x$.
7. Acceleration -8, 0, 8.
8. Acceleration 0, $-\frac{3\pi^2}{4}$, 0, $\frac{3\pi^2}{4}$, 0
9. *a.* $\frac{d^2x}{dt^2} = -4x$.
b. $\frac{d^2x}{dt^2} = -k^2x$.
c. $\frac{d^2x}{dt^2} = -\frac{x}{4}$.
d. $\frac{d^2x}{dt^2} = -k^2x$.
10. 109.47° (109° 28′).
11. 90°, 26.57° (26° 34′).

12. *a.* Max. pt. $\left(\frac{2\pi}{3}, 3.8265\right)$; min. pt. $\left(\frac{4\pi}{3}, 2.4567\right)$; infl. pts. $(0, 0)$, (π, π), $(2\pi, 2\pi)$.

c. Max. pt. $\left(\frac{\pi}{3}, 1.9132\right)$; min. pt. $\left(\frac{2\pi}{3}, 1.2284\right)$; infl. pts. $(0, 0)$, $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, (π, π).

e. Min. pt. $\left(\frac{\pi}{3}, -0.3425\right)$; max. pt. $\left(\frac{5\pi}{3}, 3.4841\right)$; infl. pts. $(0, 0)$, $\left(\pi, \frac{\pi}{2}\right)$, $(2\pi, \pi)$.

f. Max. pt. $\left(\frac{\pi}{4}, 1.4142\right)$; min. pt. $\left(\frac{5\pi}{4}, -1.4142\right)$; infl. pts. $\left(\frac{3\pi}{4}, 0\right)$, $\left(\frac{7\pi}{4}, 0\right)$.

13. 0.00029.

14. *a.* 0.8835. *b.* 0.4849. *c.* 0.8573. *d.* 0.5302.

15. $y = \sqrt{25 - 24\cos\theta}$, 3.606 ft., 0.0025 ft.

Pages 398, 399

1. *a.* $-\frac{1}{2}\cos 2x + C$.
   c. $\sin x - \cos x + C$.
   e. $\log \sin x + C$.
   g. $-e^{\cos x} + C$.
   h. $\sin \log x + C$.
   j. $-\sin(\pi - x) + C$.

2. *a.* 2.
   b. $\frac{2}{\pi}$.
   c. 1.2081.
   d. 0.6931.

3. *a.* 4.
   b. $\frac{8}{\pi}$.
   c. 1.
   d. 4.

4. $\frac{\pi^2}{2}$.

5. 3.829.

6. 14.685.

7. $\sqrt{2} - 1$.

8. $2\sqrt{2}$.

9. $2 + \frac{4}{\pi}$.

10. $4\sqrt{2}$.

11. -8.

Pages 403, 404

3. *a.* $f'(x) = a\sec^2 ax$.
   c. $\frac{ds}{dt} = -2t\csc^2 t^2$.
   e. $y' = \frac{1}{x^2}\csc\frac{1}{x}\cot\frac{1}{x}$.
   g. $F'(\theta) = \tan^2\theta$.
   h. $f'(x) = \frac{2}{\sin 2x}$.
   i. $f'(x) = \frac{1}{x}\sec^2(\log x)$.

4. *a.* $\frac{\pi}{2}$. *b.* $1 - \frac{\pi}{2}$. *c.* 4.981. *d.* $\frac{2}{3}\sqrt{3}$.

9. 53.13° (53° 8′).

10. Max. $\frac{\pi}{2} - 1$, min. $\frac{3\pi}{2} + 1$.

11. $-1200\,\pi r$ ft./min., 0.

14. 11.22 ft.

15. 12 ft.

17. 5π mi./min.

Pages 406, 407

1. *a.* $\frac{1}{a}\log\sec ax + C.$
   c. $\frac{1}{2}\sec 2t + C.$
   e. $-\frac{1}{a}\cot ax + C.$
   g. $\frac{1}{2}\tan x^2 + C.$
   i. $\tan\theta - \cot\theta + C.$
   k. $2(\tan x + \sec x) - x + C.$

2. *a.* $\frac{2}{3}\sqrt{3}.$ *b.* $\frac{4}{\pi}.$ *c.* 0.0719. *d.* $\frac{6}{\pi}\log 2.$
3. $\tan\theta + \log\sec\theta + 5.$
5. $\frac{2}{\pi}\log 2.$
7. 4.
8. 1.317.
9. 2.059.
10. $\pi - \frac{\pi^2}{4}.$

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2. *a.* $x^2 + y^2 = r^2.$
   b. $x^2 - 2x + y^2 = 0.$
   c. $9x^2 + 25y^2 = 225.$
   d. $\frac{x^2}{9} + \frac{(y-5)^2}{25} = 1.$
   e. $x^2 - y^2 = 1.$
4. $x = r\cos 2\pi t,\ y = r\sin 2\pi t;$
   $x = r\cos 6\pi t,\ y = r\sin 6\pi t.$
5. $45^\circ.$
6. $-\frac{b}{a}\cot\theta.$
8. $-\frac{1}{\sqrt{3}}.$
9. $v = 10,\ \frac{5}{2}\sqrt{10},\ 5;$
   $\alpha = 0^\circ,\ 153.43^\circ\,(153^\circ\,26'),\ 90^\circ.$
10. $v = 6,\ \alpha = 120^\circ.$
12. $v = 20.$
13. $x = 2(\pi t - \sin\pi t),$
    $y = 2(1 - \cos\pi t).$
14. 2π ft./sec.
15. $2\pi\sqrt{2}$ ft./sec.
17. $\pi ab.$
18. $3\pi a^2.$
19. $8a.$

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1. *a.* $\frac{1}{3}\sin^3 x + C.$
   b. $-\frac{1}{2}\cos^2 x + C.$
   c. $-\frac{1}{3}\cos^3 x + C.$
   d. $-\frac{1}{8}\cos^4 2\theta + C.$
   e. $\sec x + \cos x + C.$
   f. $\frac{1}{2}\tan^2 x + \log\cos x + C.$
   g. $-\cot x - x + C.$
   h. $-\frac{1}{2}\cot^2 x - \log\sin x + C.$
   i. $-\frac{1}{6}\cot^6 x - \frac{1}{4}\cot^4 x + C.$
   j. $\frac{1}{2}(\tan^2 t - \cot^2 t) + 2\log\tan t + C.$
   k. $\frac{1}{12}.$
2. *a.* $\frac{4}{3}\pi ab^2.$ *b.* $\frac{4}{3}\pi a^2 b.$
3. $\frac{3}{8}\pi a^2.$
4. $\frac{32\pi a^3}{105}.$

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1. *a.* $x \sin x + \cos x + C.$
   b. $\frac{e^{3x}}{9}(3x - 1) + C.$
   c. $-\frac{e^{-2x}}{4}(2x + 1) + C.$
   d. $\frac{1}{4}\sin 2x - \frac{x}{2}\cos 2x + C.$
   e. $x \tan x - \frac{x^2}{2} - \log \sec x + C.$
   f. $\frac{x^2}{4}(2 \log x - 1) + C.$
   g. $x(\log x - 1) + C.$
   h. $\frac{x^{n+1}}{n+1}\log x - \frac{x^{n+1}}{(n+1)^2} + C.$
   i. $-x^2 \cos x + 2x \sin x + 2 \cos x + C.$
   j. $\frac{x^2}{2}\sin 2x - \frac{1}{4}\sin 2x + \frac{x}{2}\cos 2x + C.$

2. 14.026.

3. $\left(\frac{\pi}{2}, \frac{\pi}{8}\right).$

4. 164.8.

5. $18631\,\pi.$

Pages 419, 420

1. *a.* $\arcsin \frac{x}{a} + C.$
   b. $\log\left(x + \sqrt{x^2 + a^2}\right) + C.$
   c. $\frac{1}{a}\operatorname{arc\,sec} \frac{x}{a} + C.$
   d. $\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin \frac{x}{a} + C.$
   e. $\frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left(x + \sqrt{x^2 + a^2}\right) + C.$
   f. $\pi/3.$
   g. $\frac{162}{5}.$
2. $\pi ab.$
3. $3\,\pi a^2.$
4. $5\,\pi^2 a^3.$
5. $\frac{p}{2}\left[\sqrt{2} + \log\left(1 + \sqrt{2}\right)\right].$
6. $\pi^2/2.$
7. $2\,a^2\sqrt{3} - a^2 \log\left(2 + \sqrt{3}\right).$
8. $\frac{7}{12}\sqrt{2} + \frac{1}{4}\log\left(1 + \sqrt{2}\right).$
9. $4000\,\pi$ lb.
10. $\frac{\pi r^2}{2} - a\sqrt{r^2 - a^2} - r^2 \arcsin \frac{a}{r}.$
11. 57.28.
12. *a.* 0.8814.
    b. 28.27.

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